Impulsive semilinear evolution differential inclusions with nonconvex right hand side

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Abstract: In this paper we prove the existence of a mild solution for a class of impulsive semilinear evolution differential inclusions with infinite delay in a separable Banach space.

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1 Introduction

In this paper, we are concerned by the existence of mild solutions of impulsive semilinear functional differential inclusions with infinite delay in a separable Banach space $E$. More precisely, we consider the following class of semilinear impulsive differential inclusions:

\[ x'(t) \in A(t)x(t) + F(t, x_t), \quad t \in J = [0, b], \quad t \neq t_k, \]
\[ \Delta x \big|_{t=t_k} \in I_k(x(t_k^-)), \quad k = 1, \ldots, m \]
\[ x(t) = \phi(t), \quad t \in (0, 0], \]

where $\{A(t) : t \in J\}$ is a family of linear operators in Banach space $E$ generating an evolution operator, $F$ be a lower semicontinuous multifunction from $J \times B$ to the collection of all nonempty closed compact subset of $E$, $B$ is the phase space defined axiomatically (see section 2) which contains the mapping from $(-\infty, 0]$ into $E$, $\phi \in B$, $0 = t_0 < t_1 < \ldots < t_m < t_{m+1} = b$, $I_k : E \rightarrow P(E)$, $k = 1, \ldots, m$ are multivalued maps with closed and bounded values, $x(t_k^+) = \lim_{h \to 0^+} x(t_k + h)$ and $x(t_k^-) = \lim_{h \to 0^-} x(t_k + h)$ represent the right and left limits of $x(t)$ at $t = t_k$. Finally $P(E)$ denotes the family of nonempty subsets of $E$.

The theory of impulsive differential equations has become an important area of investigation in recent years, stimulated by the numerous applications to problems arising in mechanics, electrical engineering, medicine, biology, ecology, population
dynamics, etc. During the last few decades there have been significant developments in impulse theory, especially in the area of impulsive differential equations and inclusions with fixed moments; see the monographs of Bainov and Simeonov [6], Benchohra et al. [7], Lakshmikantham et al. [23], Samoilenko and Perestyuk [26], and the references therein. For the case where the impulses are absent (i.e. \( I_k = 0, k = 1, \ldots, m \)) and \( F \) is a single-valued or multivalued map and \( A \) is a densely defined linear operator generating a \( C_0 \)-semigroup of bounded linear operators and the state space is \( C([-r, 0], E) \) or \( E \), the problem (1.1)–(1.3) has been investigated on compact intervals in, for instance, the monographs by Ahmed [4], Hale and Verduyn Lunel [18], Hu and Papageorgiou [20], Kamenskii et al. [21] and Wu [27] and the paper of Benchohra and Ntouyas [8]. Recently some existence results were obtained for certain classes of functional differential equations and inclusions in Banach spaces under the assumption that the linear part generates a compact semigroup (see, e.g., [1, 2, 3]).

Our goal here is to give existence results for the problem (1.1)–(1.3), in the case where the set-valued maps is lower semicontinuous. We use Mönch’s fixed point theorem combined with a selection theorem due to Bressan and Colombo (see [11]). In Section 2, we will recall briefly some basic definitions and preliminary facts which will be used throughout the following sections. In Section 3, we prove existence of solutions for problem (1.1)–(1.3).

We mention that the model with multivalued jump sizes may arise in a control problem where we want to control the jump sizes in order to achieve given objectives [5]. To our knowledge, there are very few results for impulsive evolution inclusions with multivalued jump operators; see [3, 9, 10, 25]. The results of the present paper extend and complement those obtained in the absence of the impulse functions \( I_k \), and for those with single-valued impulse functions \( I_k \).

### 2 Preliminaries

In this section, we introduce notations, definitions, and preliminary facts which are used throughout this paper.

Let \( J = [0, b], b > 0 \), \((E, \| \cdot \|)\) be a real separable Banach space, \( C(J, E) \) the space of \( E \)-valued continuous functions on \( J \) with the uniform norm

\[
\|x\| = \sup \{\|x(t)\|, \ t \in J\},
\]

\( L^1(J, E) \) the space of \( E \)-valued Bochner integrable functions on \( J \) with the norm

\[
\|f\|_{L^1} = \int_0^b \|f(t)\|dt.
\]
To define the solution of problem (1.1)–(1.3), it is convenient to introduce some additional concepts and notations. Consider the following spaces
\[ PC(J, E) = \{ y : [0, b] \to E, y_k \in C(J_k; E) \text{ where } y(t_k^+), y(t_k^-) \text{ with } y(t_k^-) = y(t_k^+) \} \]
where \( y_k \) is the restriction of \( y \) to \( J_k = (t_k, t_{k+1}], k = 0, \ldots, m. \)

\[ \Omega = \{ y \in (-\infty, b] \to E : \ y|_{(-\infty,0]} \in B \text{ and } y|_{[0,b]} \in PC(J, E) \} \]
be the semi-norm in \( \Omega \) defined by
\[ \| y \|_{\Omega} = \| y_0 \|_B + \sup \{ \| y(s) \| : 0 \leq s \leq b \}, \ y \in PC. \]

In this work, we will employ an axiomatic definition for the phase space \( B \) which is similar to those introduced in [19]. Specifically, \( B \) will be a linear space of functions mapping \( (-\infty, 0] \) into \( E \) endowed with a semi norm \( \| \cdot \|_1 \), and satisfies the following axioms introduced at first by Hale and Kato in [17]:

(A1) There exist a positive constant \( H \) and functions \( K(\cdot), M(\cdot) : \mathbb{R}^+ \to \mathbb{R}^+ \) with \( K \) continuous and \( M \) locally bounded, such that for any \( b > 0 \) if \( y : (-\infty, b] \to E \), such that \( y|_{[0,b]} \in PC(J, E) \) and \( y_0 \in B \); the following conditions hold:

(i) \( y_t \) is in \( B \);
(ii) \( \| y(t) \| \leq H \| y_t \|_B \);
(iii) \( \| y_t \|_B \leq K(t) \sup \{ \| y(s) \| : 0 \leq s \leq b \} + M(t) \| y_0 \|_B \) and \( H, K \) and \( M \) are independent of \( y(\cdot) \).

(A2) The space \( B \) is complete.

In what follows we use the following notations \( K_b = \sup \{ K(t), t \in J \} \) and \( M_b = \sup \{ M(t), t \in J \} \)

**Definition 2.1.** Let \( X \) and \( Y \) be two topological vector spaces. We denote by \( P(Y) \) the family of all non-empty subsets of \( Y \) and by

\[ P_k(Y) = \{ C \in P(Y) : \text{compact} \}, \ P_b(Y) = \{ C \in P(Y) : \text{bounded} \}, \]
\[ P_c(Y) = \{ C \in P(Y) : \text{closed} \}, \ P_{cv}(Y) = \{ C \in P(Y) : \text{convex} \}. \]

A multifunction \( G : X \to P(Y) \) is said to be lower semicontinuous (l.s.c.) if \( G^{-1}(V) = \{ x \in X : G(x) \cap V \neq \emptyset \} \) is an open subset of \( X \) for every open \( V \subseteq Y \). The multifunction \( G \) is called closed if its graph \( \Gamma_G = \{ (x, y) \in X \times Y : y \in G(x) \} \) is closed subset of the topological space \( X \times Y \). A multifunction \( \mathcal{F} : [c, d] \subset \mathbb{R} \to P_k(Y) \)
is said to be strongly measurable if there exists a sequence $F_n : [c, d] \to P_k(Y),
\quad n = 1, 2, \ldots$ of steps multifunctions such that
\[
\lim_{n \to +\infty} h(F_n(t), F(t)) = 0, \quad \text{for } \mu\text{-a.e } t \in [c, d],
\]
where $\mu$ denotes the Lebesgue measure on $[c, d]$ and $h$ is the Hausdorff metric on $P_k(Y)$.

A subset $B$ of $L^1([0, T]; E)$ is decomposable if for all $u(.); v(.) \in B$ and $I \subset [0, T]$ measurable, the function $u(.)X_I + v(.)X_{[0,T]} \in B$, where $X$ denotes the characteristic function.

**Definition 2.2.** Let $F : [0, T] \to P(E)$ be a multi-valued map with nonempty compact values. Assign to $F$ the multi-valued operator
\[
F : C([0, T]; E) \to P(L^1([0; T]; E)),
\]
defined by
\[
F(x(.)) = \{y(.) \in L^1([0, T]; E) : y(t) \in F(t; x(t)), \text{ for a.e. } t \in [0, T]\}.
\]
The operator $F$ is called the Niemytzki operator associated with $F$. We say $F$ is the lower semi-continuous type if its associated Niemytzki operator $F$ is lower semi-continuous and has nonempty closed and decomposable values. For details and equivalent definitions see [15, 21, 22].

Let us recall the following result that will be used in the sequel.

**Lemma 2.3.** [11] Let $E$ be a separable metric space and let $G : E \to P(L^1([0, T]; E))$ be a multi-valued operator which is lower semi-continuous and has nonempty closed and decomposable values. Then $G$ has a continuous selection, i.e. there exists a continuous function $f : E \to L^1([0, T]; E)$ such that $f(y) \in G(y)$ for every $y \in E$.

**Definition 2.4.** Let $(A, \geq)$ be a partially ordered set. A function $\beta : P_b(E) \to A$ is called a measure of noncompactness (MNC) in $E$ if
\[
\beta(\overline{\omega}\Omega) = \beta(\Omega), \quad \text{for every } \Omega \in P_b(E).
\]

**Definition 2.5.** A measure of noncompactness $\beta$ is called:

(i) monotone if $\Omega_0, \Omega_1 \in P_b(E), \Omega_0 \subset \Omega_1$ implies $\beta(\Omega_0) \leq \beta(\Omega_1)$

(ii) nonsingular if $\beta(\{a\} \cup \Omega) = \beta(\Omega)$ for every $a \in E, \Omega \in P_b(E)$;

(iii) regular if $\beta(\Omega) = 0$ is equivalent to the relative compactness of $\Omega$. 
As an example of the measure of noncompactness possessing all these properties is the Hausdorff of MNC which is defined by
\[
\chi(\Omega) = \inf\{\varepsilon > 0 : \Omega \text{ has a finite } \varepsilon \text{ -- net}\}.
\]
For more information about the measure of noncompactness we refer the reader to [21].

**Definition 2.6.** A countable set \( \{f_n : n \geq 1\} \subseteq L^1(J, E) \) is said to be semicompact if

(i) it is integrably bounded: \( \|f_n(t)\| \leq \omega(t) \) for a.e. \( t \in J \) and every \( n \geq 1 \) where \( \omega \in L^1(J, \mathbb{R}^+) \)

(ii) the set \( \{f_n(t) : n \geq 1\} \) is relatively compact in \( E \) for a.e. \( t \in J \).

Now, let for every \( t \in J \), \( A(t) : E \rightarrow E \) be a linear operator such that

(i) For all \( t \in J \), \( D(A(t)) = D(A) \subseteq E \) is independent of \( t \).

(ii) For each \( s \in I \) and each \( x \in E \) there is a unique solution \( v : [s, b] \rightarrow E \) for the evolution equation
\[
v'(t) = A(t)v(t), \quad t \in [s, b] \\
v(s) = x.
\]

(2.1)

In this case an operator \( T \) can be defined as
\[
T : \Delta = \{(t, s) : 0 \leq s \leq t \leq b\} \rightarrow \mathcal{L}(E), \quad T(t, s)(x) = v(t),
\]
where \( v \) is the unique solution of (2.1) and \( \mathcal{L}(E) \) is the family of linear bounded operators on \( E \).

**Definition 2.7.** The operator \( T \) is called the evolution operator generated by the family \( \{A(t) : t \in J\} \).

1. \( T(s, s) = I_E \),

2. \( T(t, r)T(r, s) = T(t, s) \), for all \( 0 \leq s \leq r \leq t \leq b \).

3. \( t \rightarrow T(t, s) \) is strongly continuous and
\[
\frac{\partial T(t, s)}{\partial t} = A(t)T(t, s), \quad \frac{\partial T(t, s)}{\partial s} = -T(t, s)A(s).
\]
**Definition 2.8.** The operator $G : L^1(J, X) \to C(J, X)$ defined by

$$Gf(t) = \int_0^t T(t, s)f(s)ds$$

is called the generalized Cauchy operator, where $T(.,.)$ is the evolution operator generated by the family of operators $\{A(t) : t \in J\}$.

In the sequel we will need the following results.

**Lemma 2.9.** [21] Every semicompact set in $L^1(J, E)$ is weakly compact in the space $L^1(J, E)$.

**Lemma 2.10** ([21, Theorem 2]). The generalized Cauchy operator $G$ satisfies the properties

(1) there exists $\zeta \geq 0$ such that

$$\|Gf(t) - Gg(t)\| \leq \zeta \int_0^t \|f(s) - g(s)\|ds,$$

for every $f, g \in L^1(J, E)$, $t \in J$.

(2) for any compact $K \subseteq E$ and sequence $(f_n)_{n \geq 1}$, $f_n \in L^1(J, E)$ such that for all $n \geq 1$, $f_n(t) \in K$, a. e. $t \in J$, the weak convergence $f_n \rightharpoonup f_0$ in $L^1(J, E)$ implies the convergence $Gf_n \to Gf_0$ in $C(J, E)$.

**Lemma 2.11.** [21] Let $S : L^1(J, E) \to C(J, E)$ be an operator satisfying condition (1) and the following Lipschitz condition (weaker than (1)).

(1')

$$\|Sf - Sg\|_{C(J, E)} \leq \zeta \|f - g\|_{L^1(J, E)}.$$

Then for every semicompact set $\{f_n\}_{n=1}^{+\infty} \subseteq L^1(J, E)$ the set $\{Sf_n\}_{n=1}^{+\infty}$ is relatively compact in $C(J, E)$. Moreover, if $(f_n)_{n \geq 1}$ converges weakly to $f_0$ in $L^1(J, E)$ then $Sf_n \to Sf_0$ in $C(J, E)$.

**Lemma 2.12.** [21] Let $S : L^1(J, E) \to C(J, E)$ be an operator satisfying conditions (G1), (G2) and let the set $\{f_n\}_{n=1}^{+\infty}$ be integrably bounded with the property $\chi(\{f_n(t) : n \geq 1\}) \leq \eta(t)$, for a.e. $t \in J$, where $\eta(.) \in L^1(J, \mathbb{R}^+)$ and $\chi$ is the Hausdorff MNC. Then

$$\chi(\{Sf_n(t) : n \geq 1\}) \leq 2\zeta \int_0^t \eta(s)ds$$

for all $t \in J$, where $\zeta \geq 0$ is the constant in condition (G1).
Theorem 2.13. [24] Let $E$ be a Banach space, $U$ an open subset of $E$ and $0 \in U$. Suppose that $N : U \to E$ is a continuous map which satisfies Mönch’s condition (that is, if $D \subseteq \overline{U}$ is countable and $D \subseteq \overline{\partial}(\{0\} \cup N(D))$, then $D$ is compact) and assume that 

$$x \neq \lambda N(x), \quad \text{for } x \in \partial U \text{ and } \lambda \in (0, 1)$$

holds. Then $N$ has a fixed point in $\overline{U}$.

3 Existence Theorem

In this section we prove the existence of mild solutions for the impulsive semilinear functional differential inclusions (1.1)–(1.3).

We will assume the following hypothesis

(A) \{\{A(t) : t \in J\} be a family of linear (not necessarily bounded) operators, A(t) : D(A) \subset E \to E, D(A) not depending on $t$ and dense subset of $E$ and T : $\Delta = \{(t, s) : 0 \leq s \leq t \leq b\} \to \mathcal{L}(E)$ be the evolution operator generated by the family \{A(t) : t \in J\}. Let $F$ be a multifunction defined from $J \times \mathcal{B}$ to the family of nonempty closed convex subsets of $E$ such that

(H1) \((t, x) \mapsto F(., x)\) is $\mathcal{L} \otimes \mathcal{B}_b$ measurable ($\mathcal{B}_b$ is Borel measurable).

(H2) The multifunction $F : (t, .) \to P_k(E)$ is lower semicontinuous for a.e. $t \in J$.

(H3) there exists a function $\alpha \in L^1(J, \mathbb{R}^+)$ such that 

$$\|F(t, \psi)\| \leq \alpha(t), \quad \text{for a.e. } t \in J, \forall \psi \in \mathcal{B};$$

(H4) There exists a function $\beta \in L^1(J, \mathbb{R}^+)$ such that for all $D \subset \mathcal{B}$, we have 

$$\chi(F(t, D)) \leq \beta(t) \sup_{-\infty \leq s \leq 0} \chi(D(s)) \quad \text{for a.e. } t \in J,$$

where, $D(s) = \{x(s); x \in D\}$ and $\chi$ is the Hausdorff measure of noncompactness.

(H5) There exist constants $a_k, b_k \geq 0, k = 1, \ldots, m$ with $M \sum_{k=1}^{k=m} |a_k| < 1$ such that 

$$\|I_k\| \leq a_k\|x\| + b_k, \quad \text{where } I_k \in I_k(x(t^+_k)).$$

(H6) There exist constants $c_k \geq 0, k = 1, \ldots, m$, such that 

$$\chi(I_k(D)) \leq c_k\chi(I_k(D)).$$
**Definition 3.1.** A function $x \in \Omega$ is said to be a mild solution of system (1.1)–(1.3) if there exist a function $f \in L^1(J; E)$ such that $f \in F(t, x_t)$ for a.e. $t \in J$ and $I_k \in I_k(t_k^+))$

(i) $x(t) = T(t, 0)\phi(0) + \int_0^t T(t, s)f(s)ds + \sum_{0< t_k< t}T(t, t_k)I_k \ , \ \text{a.e. } t \in J, \ k = 1, \ldots, m$

(ii) $x(t) = \phi(t), \ t \in (-\infty, 0]$.

**Remark 3.2.** Under conditions (H1)-(H3) for every piecewise continuous function $v : [0, b] \to B$ the multifunction $F(t, v(t))$ admits a Bochner integrable selection (see [21]).

Now we state and prove our main result.

**Theorem 3.3.** Under assumptions (A) and (H1)-(H6), the problem (1.1)–(1.3) has at least one mild solution.

**Proof.** We note that from assumptions (F1) and (F3) it follows that the superposition multioperator $S_{F_1}^1 : \Omega \to \mathcal{P}(L^1(J, E))$, defined by

$$S_{F_1}^1(x) = \{f \in L^1(J, E) : f(t) \in F(t, x_t), \ \text{a.e. } t \in J\}$$

is nonempty set (see [21]).

**Step 1.** The Mönch’s condition holds.

Suppose that $\Omega \subseteq B_r$ is countable and $\Omega \subseteq \overline{B}(\{0\} \cup N(\Omega))$. We will prove that $\Omega$ is relatively compact.

We consider the measure of noncompactness defined in the following way. For every bounded subset $\Omega \subset \Omega$

$$\nu_1(\Omega) = \max_{D \in \Delta(\Omega)} (\gamma_1(D), \ \text{mod} C(D)) \in \mathbb{R}^2, \quad (3.1)$$

where $\Delta(\Omega)$ is the collection of all the denumerable subsets of $\Omega$;

$$\gamma_1(D) = \sup_{t \in J} e^{-Lt} \chi(\{x(t) : x \in D\}); \quad (3.2)$$

where $\text{mod} C(D)$ is the modulus of equicontinuity of the set of functions $D$ given by the formula

$$\text{mod} C(D) = \lim_{\delta \to 0} \sup_{x \in D} \max_{|t_1 - t_2| \leq \delta} \|x(t_1) - x(t_2)\|; \quad (3.3)$$
and $L > 0$ is a positive real number chosen so that

$$q := M \left( 2 \sup_{t \in J} \int_{0}^{t} e^{-L(t-s)} \beta(s)ds + e^{L t} \sum_{k=0}^{m} c_k \right) < 1 \quad (3.4)$$

where $M = \sup_{(t,s) \in \Delta} \|T(t,s)\|$.

From the Arzela-Ascoli theorem, the measure $\nu_1$ give a nonsingular and regular measure of noncompactness, (see [21]).

Let $\{y_n\}_{n=1}^{+\infty}$ be the denumerable set which achieves that maximum $\nu_1(N(\Omega))$, i.e;

$$\nu_1(N(\Omega)) = (\gamma_1(\{y_n\}_{n=1}^{+\infty}), \mod_{C(\{y_n\}_{n=1}^{+\infty})}).$$

Then there exists a set $\{x_n\}_{n=1}^{+\infty} \subset \Omega$ such that $y_n \in \gamma_1(x_n)$, $n \geq 1$. Then

$$y_n(t) = T(t,0)\phi(0) + \int_{0}^{t} T(t,s)f(s)ds + \sum_{0 < t_k < t} T(t,t_k)I_k, \quad (3.5)$$

where $f \in S^{1}_{F}$ and $I_k \in I_k(x)$, so that

$$\gamma_1(\{y_n\}_{n=1}^{+\infty}) = \gamma_1(\{Gf_n\}_{n=1}^{+\infty}).$$

We give an upper estimate for $\gamma_1(\{y_n\}_{n=1}^{+\infty})$.

Fixed $t \in J$ by using condition (H4), for all $s \in [0,t]$ we have

$$\chi(\{f_n(s)\}_{n=1}^{+\infty}) \leq \chi(F(s,\{x_n(s)\}_{n=1}^{+\infty}))$$

$$\quad \leq \beta(s)\chi(\{x_n(s)\}_{n=1}^{+\infty})$$

$$\quad \leq \beta(s)e^{Ls} \sup_{t \in J} e^{-Lt} \chi(\{x_n(t)\}_{n=1}^{+\infty})$$

$$\quad = \beta(s)e^{Ls}\gamma_1(\{x_n\}_{n=1}^{+\infty}).$$

By using condition (H3), the set $\{f_n\}_{n=1}^{+\infty}$ is integrably bounded. In fact, for every $t \in J$, we have

$$\|f_n(t)\| \leq \|F(t,x_n(t))\|$$

$$\quad \leq \alpha(t).$$

By applying Lemma 2.12, it follows that

$$\chi(\{Gf_n(s)\}_{n=1}^{+\infty}) \leq 2M \int_{0}^{s} \beta(t)e^{Lt} (\gamma_1(\{x_n\}_{n=1}^{+\infty}))dt$$

$$\quad = 2M \gamma_1(\{x_n\}_{n=1}^{+\infty}) \int_{0}^{s} \beta(t)e^{Lt} dt.$$
Thus, we get
\[
\gamma_1(\{x_n\}_{n=1}^{+\infty}) \leq \gamma_1(\{y_n\}_{n=1}^{+\infty}) \\
= \sup_{t \in J} e^{-Lt} 2M \gamma_1(\{x_n\}_{n=1}^{+\infty}) \int_0^t \beta(s) e^{Ls} ds + M \gamma_1(\{x_n\}_{n=1}^{+\infty}) e^{Lt} \sum_{k=1}^m c_k \\
\leq q \gamma_1(\{x_n\}_{n=1}^{+\infty}).
\] (3.6)

Therefore, we have that
\[
\gamma_1(\{x_n\}_{n=1}^{+\infty}) \leq \gamma_1(\Omega) \leq \gamma_1(\{0\} \cup N(\Omega)) \gamma_1(\{y_n\}_{n=1}^{+\infty}) \leq q \gamma_1(\{x_n\}_{n=1}^{+\infty}).
\]

From (3.4), we obtain that
\[
\gamma_1(\{x_n\}_{n=1}^{+\infty}) = \gamma_1(\Omega) = \gamma_1(\{y_n\}_{n=1}^{+\infty}).
\]

Coming back to the definition of \(\gamma_1\), we can see
\[
\chi(\{x_n\}_{n=1}^{+\infty}) = \chi(\{y_n\}_{n=1}^{+\infty}) = 0.
\]

By using the last equality and hypotheses (H3) and (H4) we can prove that set \(\{f_n\}_{n=1}^{+\infty}\) is semicompact. Now, by applying Lemma 2.10 and Lemma 2.11, we can conclude that set \(\{Gf_n\}_{n=1}^{+\infty}\) is relatively compact in \(C(J; E)\).

The representation of \(y_n\) given by (3.5) yields that set \(\{y_n\}_{n=1}^{+\infty}\) is also relatively compact in \(C(J; E)\), since \(\nu_1\) is a monotone, nonsingular, regular MNC, we have that
\[
\nu_1(\Omega) \leq \nu_1(\overline{0} \cup N(\Omega)) \leq \nu_1(N(\Omega)) = \nu_1(\{y_n\}_{n=1}^{+\infty}) = (0, 0).
\]

Therefore, \(\Omega\) is relatively compact.

**Step 2.** It is clear that the superposition multioperator \(S_1^F\) has closed and decomposable values. Following the lines of [21], we may verify that \(S_1^F\) is l.s.c. Applying Lemma 2.3 to the restriction of \(S_1^F\) on \(\Omega\) we obtain that there exists a continuous selection \(w : \Omega \rightarrow L^1(J, E)\). We consider a map \(N_1 : \Omega \rightarrow \Omega\) defined as
\[
x(t) = T(t, 0)\phi(0) + \int_0^t T(t, s) w(x)(s) ds.
\]

Since the Cauchy operator is continuous, the map \(N_1\) is also continuous, therefore, it is a continuous selection of the integral multioperator.

**Step 3.** A priori bounds.
We will demonstrate that the solutions set is a priori bounded. Indeed, let $x \in \lambda N_1$ and $\lambda \in (0, 1)$. There exists $f \in S^1_F$ and $I_k \in \mathcal{I}_k(x)$ such that for every $t \in J$ we have

$$
\|x(t)\| = \|\lambda T(t, 0)\phi(0) + \lambda \int_0^t T(t, s)f(s)ds + \lambda \sum_{0 < t_k < t} T(t, t_k)I_k\| \\
\leq M(\|\phi(0)\| + \|x\| \sum_{k=1}^m \|a_k\| + \sum_{k=1}^m \|b_k\|) + M \int_0^t \alpha(s)ds
$$

hence,

$$(1 - M \sum_{k=1}^m \|a_k\|)\|x\| \leq M(\|\phi(0)\| + \|\alpha\| + \sum_{k=1}^m \|b_k\|).$$

Consequently

$$\|x\| \leq \frac{M(\|\phi(0)\| + \|\alpha\| + \sum_{k=1}^m \|b_k\|)}{1 - M \sum_{k=1}^m \|a_k\|} = C.$$ 

So, there exists $N^*$ such that $\|x\| \neq N^*$, set

$$U = \{x \in \Omega : \|x\| < N^*\}.$$ 

From the choice of $U$ there is no $x \in \partial U$ such that $x = \lambda N_1 x$ for some $\lambda \in (0, 1)$.

Thus, we get a fixed point of $N_1$ in $\bar{U}$ due to the Mönch’s Theorem.

\[\square\]

4 An example

As an application of our results we consider the following impulsive partial functional differential equation of the form

$$\frac{\partial}{\partial t} z(t, x) + a(t, x) \frac{\partial^2}{\partial x^2} z(t, x) + \int_{-\infty}^0 P(\theta)r(t, z(t + \theta, x))d\theta, \quad x \in [0, \pi], \ t \in [0, b], \ t \neq t_k,$$

$$z(t_k^+, x) - z(t_k^-, x) \in [-b_k|z(t_k^-, x), b_k|z(t_k^-, x)|, \ x \in [0, \pi], \ k = 1, \ldots, m, \ (4.2)$$

$$z(t, 0) = z(t, \pi), \quad t \in J := [0, b], \ (4.3)$$

$$z(t, x) = \phi(t, x), \quad -\infty < t \leq 0, \ x \in [0, \pi], \ (4.4)$$

where $a(t, x)$ is continuous function and uniformly Hölder continuous in $t$, $b_k > 0$, $k = 1, \ldots, m$, $\phi \in \mathcal{D},$

$$\mathcal{D} = \{\overline{\psi} : (-\infty, 0] \times [0, \pi] \to \mathbb{R}; \ \overline{\psi} \text{ is continuous everywhere except for a countable number of points at which } \overline{\psi}(s^-), \overline{\psi}(s^+) \text{ exist with } \overline{\psi}(s^-) = \overline{\psi}(s)\},$$
0 = t_0 < t_1 < t_2 < ... < t_m < t_{m+1} = b, \ z(t_k^+) = \lim_{(h,x) \to (0^+,x)} z(t_k + h, x), \\
z(t_k^-) = \lim_{(h,x) \to (0^-,x)} z(t_k + h, x), \ P : (-\infty,0] \to \mathbb{R} \text{ a continuous function, } r : \mathbb{R} \times \mathbb{R} \to \mathcal{P}_k(\mathbb{R}) \text{ a lower semicontinuous multivalued map.}

Let
\[
y(t) = z(t,x), \quad x \in [0,\pi], \ t \in J = [0,b],
\]
\[
\mathcal{I}_k(y(t_k^-))(x) = [-b_k|z(t_k^-,x),b_k|z(t_k^-,x)], \quad x \in [0,\pi], \ k = 1,\ldots,m,
\]
\[
F(t,\phi)(x) = \int_{-\infty}^{0} P(\theta)r(t,z(t+\theta,x))d\theta
\]
\[
\phi(\theta)(x) = \phi(\theta,x), \quad -\infty < t \leq 0, \ x \in [0,\pi].
\]

Consider \( E = L^2[0,\pi] \) and define \( A(t) \) by \( A(t)w = a(t,x)w'' \) with domain
\[
D(A) = \{w \in E : w, w' \text{ are absolutely continuous, } w'' \in E, \ w(0) = w(\pi) = 0\}.
\]

Then \( A(t) \) generates an evolution system \( U(t,s) \) satisfying assumption (H1) and (H3) (see [14]). We can show that problem (4.1)–(4.4) is an abstract formulation of problem (1.1)–(1.3). Under suitable conditions, the problem (1.1)–(1.3) has at least one mild solution.

References


