Blow-up and continuation of solutions for some semilinear parabolic equations with nonlinear boundary conditions

Junichi Harada, Mitsuharu Ôtani

Abstract: The following semilinear parabolic equations with nonlinear boundary conditions is studied: \( u_t = \Delta u - qu^{2q-1} \) in \( B_1 \times (0, T) \), \( \partial_\nu u = |u|^{q-1}u \) on \( \partial B_1 \times (0, T) \), where \( B_1 \subset \mathbb{R}^n \) is the unit ball and \( q > 1 \). The finite time blow-up as well as a continuation of blow-up solutions beyond the blow-up time are discussed for the multidimensional case. It is revealed that the behavior of solutions for the multidimensional case differs from that of the one dimensional case.

Keywords: blow-up, continuation, nonlinear boundary conditions.

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Dedicated to the memory of Professor Riichi Iino

1 Introduction

In this paper, we are concerned with positive solutions of the following equations.

\[
\begin{cases}
   u_t = \Delta u - au^p & \text{in } \Omega \times (0, T), \\
   \partial_\nu u = u^q & \text{on } \partial \Omega \times (0, T), \\
   u(x, 0) = u_0(x) \geq 0 & \text{in } \Omega,
\end{cases}
\]  

(1.1)

where \( p, q > 1, a > 0, \Omega \) is a bounded domain in \( \mathbb{R}^n \) with smooth boundary \( \partial \Omega \) and \( \nu \) denotes the unit outer normal vector on \( \partial \Omega \). The study of the finite time blow-up and the global existence of solutions for this problem was started in \([7]\) and independently in \([19]\), and later the results were improved by many authors (see \([1], [3], [4], [8], [9], [21], [22], [23]\)), and also extended to quasilinear parabolic equations (see \([1], [10], [11], [15], [17], [24]\)). It is known that the dynamics of (1.1) is classified into three cases:

(i) \( p > \tilde{q} \) or \( p = \tilde{q}, a > q \), (ii) \( p < \tilde{q} \) or \( p = \tilde{q}, a < q \), (iii) \( p = \tilde{q}, a = q \),
where \( \tilde{q} = 2q - 1 \). For the case (i), every solution is globally defined and uniformly bounded, while for the case (ii), solutions with large initial data blow up in a finite time (see [1, 7, 23]). The critical case (iii) is also studied in [7], but only for the case \( n = 1 \). They proved that every nonnegative solution is globally defined and converges to the unique positive solution of

\[
\begin{aligned}
\Psi'' &= q\Psi^{2q-1} \quad \text{in } (-1, 1), \\
\Psi &= \infty \quad \text{on } \{1, 1\}.
\end{aligned}
\]

Namely, every nonnegative solution becomes unbounded at \( x = \pm 1 \) as \( t \to \infty \). However the critical case has not yet been studied for the multidimensional case.

The main purpose of this paper is to study the behavior of solutions of (1.1) for the critical case with \( n \geq 2 \). Our first result reveals that the behavior of solutions of (1.1) for the multidimensional case differs from that of the one dimensional case, which is stated as follows.

**Theorem 1.1.** Let \( p = 2q - 1 \) and \( a = q \). Then every positive classical solution of (1.1) blows up in a finite time if \( n \geq 2 \). In particular, if \( \Omega \) is a ball, every positive solution blows up only on the boundary.

One of our main concerns in this paper is the asymptotic behavior of blow-up solutions near the blow-up time and the continuation of blow-up solutions beyond the blow-up time. To discuss these properties, however, we here restrict ourselves to the radially symmetric case. Let \( u(r, t) \) be a radially symmetric solution of (1.1) and introduce a new unknown function \( v \) by

\[
v(r, t) = u(r, t)^{-(q-1)}.
\]

Then it is easy to see that \( v(r, t) \) solves

\[
\begin{aligned}
& v_t = v_{rr} + \frac{n-1}{r} v_r + \frac{q}{(q-1)} v \left( (q-1)^2 - v_r^2 \right) \quad \text{in } (0, R) \times (0, T), \\
& \partial_r v = -(q-1) \quad \text{on } \{R\} \times (0, T), \\
& v(r, 0) = v_0(r) := u_0(r)^{-(q-1)} \quad \text{in } (0, R).
\end{aligned}
\]

Through this transformation, blow-up phenomena of original solutions \( u(r, t) \) are equivalent to vanishing phenomena of \( v(r, t) \). More precisely, let \( u(r, t) \) be a solution of (1.1) which blows up at \( t = T \), then \( v(r, t) \) vanishes on the boundary at \( t = T \) (cf. Theorem 1.1), i.e., \( \lim_{t \to T^-} v(R, t) = 0 \). In what follows we call \( T \) ”vanishing time” of \( v \). As a consequence, the factor \( 1/v \) of the nonlinear term on the right-hand side of (1.2) may have a singularity on the boundary at \( t \sim T \). However since the factor \( (q-1)^2 - v_r^2 \) of the nonlinear term always vanishes on the boundary by virtue of
the boundary condition, we can expect that because of the cancellation, the total nonlinear term may stay bounded as far as the solution exists. This observation suggests that \( v(r,t) \) might be extended until \( t = T \) smoothly in some sense. In fact, our second result justifies this anticipation.

**Theorem 1.2.** Let \( u(r,t) \) be a positive solution of (1.1), \( T > 0 \) be its blow-up time and \( v(r,t) \) be defined above. Then \( v(r,t) \) can be continuously extended up to \( t = T \) in the sense that

\[
v \in C^{1,0}([0, R] \times [0, T]), \quad v_t, \ v_{rr} \in L^\infty((0, R) \times (0, T)).
\]

Moreover there exists \( c > 0 \) such that

\[
\|u(t)\|_\infty \geq c (T - t)^{-1/(q-1)}.
\] (1.3)

If the initial datum \( u_0(r) \) is smooth and satisfies \( u_0'(r) \leq u_0(r)^q \) in \((0, R)\) and \( u_0'(R) = u_0(R)^q \), then there exists \( \tilde{c} > 0 \) such that

\[
\|u(t)\|_\infty \leq \tilde{c} (T - t)^{-1/(q-1)}.
\] (1.4)

Finally we discuss a continuation of a solution \( v \) of (1.2) beyond its vanishing time \( T \). To construct solutions in \( t \in (T, \infty) \), we solve (1.2) in a suitable sense by regarding \( v(\cdot, T) \) as the initial datum. This way of continuation is motivated by [12]. In their paper, they considered the one dimensional nonlinear parabolic equation:

\[
\begin{cases}
    u_t = u_{xx} + e^{u_x}, & x \in (0, L), \ t > 0, \\
    u(0, t) = u(L, t) = 0, & t > 0.
\end{cases}
\]

They showed that if \( L > 0 \) is sufficiently large, then for any \( C^1 \)-initial data, there exists \( T > 0 \) such that the solution \( u(x, t) \) satisfies

\[
\lim_{t \to T^-} u_x(0, t) = +\infty.
\]

To construct a global solution beyond the blow-up time \( T \), they introduced a new unknown function \( w(x, t) = e^{-u_x(x,t)} \), which satisfies the following equation similar to (1.2):

\[
\begin{cases}
    w_t = w_{xx} + \frac{1}{w}(w_x - w_x^2), & x \in (0, L), \ t > 0, \\
    w_x(0, t) = w_x(L, t) = 1, & t > 0.
\end{cases}
\] (1.5)

They constructed a global solution \( \bar{w} \in C^{1,0}([0, R] \times [0, \infty)) \) of (1.5) satisfying \( \bar{w}(x, t) \equiv w(x, t) \) for \( t < T \) by introducing some approximation for this equation.
and applying some limiting procedure. In particular, the solution \( \bar{w}(x,t) \) satisfies \( \bar{w}(0,t) \equiv 0 \) for \( t \geq T \).

Following their method, we construct global solutions of (1.2) in a suitable class of functions \( K \). We call \( \bar{v}(r,t) \) a global solution of (1.2) in \( K \), if \( \bar{v}(r,t) \) belongs to

\[
K = \{ \bar{v} \in C^{1,0}([0,R] \times (0, + \infty)) ; \bar{v}(r,t) > 0 \text{ in } [0,R] \times (0, + \infty), \\
\int_0^R \int_0^\tau (\bar{v}_t^2 + \bar{v}_{rr}^2) r^{n-1} dr \, dt < + \infty \quad \forall \tau > 0 \}
\]

and satisfies (1.2) with \( T \) replaced by \( + \infty \), where the boundary condition is satisfied in the classical sense. The following result guarantees the existence of such a global solution.

**Theorem 1.3.** Let \( v(r,t) \) be a solution of (1.2) which vanishes at \( t = T > 0 \). Then there exists a global solution \( \bar{v}(r,t) \) of (1.2) in \( K \) such that \( \bar{v}(r,t) = v(r,t) \) for \( t \in [0,T) \) and

\[
\bar{v}(R,t) = 0, \quad t \in (T, \infty). \tag{1.6}
\]

Furthermore the uniqueness of global solutions of (1.2) holds true in the following sense.

**Theorem 1.4.** Equation (1.2) admits at most one global solution in \( K \).

The rest of this paper is organized as follows. In Section 2, we prepare the properties of zeros of solutions of linear parabolic equations and introduce singular solutions. In Section 3, we discuss a finite time blow-up and give a proof of Theorem 1.1. The existence of the blow-up profile is discussed in Section 4. The construction of a global solution of (1.2) beyond the vanishing time is discussed in Section 5 and the uniqueness of global solutions is proved in Section 6.

Throughout this paper, we denote by \( B_R \) the ball in \( \mathbb{R}^n \) centered at the origin with radius \( R \) and we frequently use the notations

\[
\tilde{q} = (q - 1), \quad \hat{q} = q/(q - 1).
\]

We denote by \( L_p^p(0,R) \) the function space of radially symmetric functions \( u(r) \) (\( r = |x| \) ) satisfying \( \|u\|_p < + \infty \), where \( \| \cdot \|_p \) is given by

\[
\|u\|_p := \begin{cases} \left( \int_0^R |u(r)|^p r^{n-1} dr \right)^{1/p} & \text{if } p \in [1, \infty), \\
\sup_{r \in (0,R)} |u(r)| & \text{if } p = \infty. \end{cases}
\]
The Sobolev spaces associated with $L^p(0, R)$ are defined by

$$W^{k,p}_t(0, R) = \{ u \in L^p_t(0, R); \| \partial^i_t u \|_p < +\infty \text{ for } i = 1, \ldots, k \}.$$ 

Let $Q^R_T = (0, R) \times (0, T)$ be the space-time domain and define the space-time $L^p$ space by

$$L^p(Q^R_T) = \{ u(r,t) \in L^1_{loc}(Q^R_T); \| u \|_{L^p(Q^R_T)}^p = \int_0^T \| u(t) \|_p^p dt < +\infty \}.$$ 

We denote the positive (negative) part of $u$ by $u^+ = \max\{0, u\}$ or $[u]^+ (u^- = \max\{0, -u\} or [u]^-)$. From this definition, it is clear that $u = u^+ - u^-$. 

## 2 Preliminaries

In this section we collect some fundamental facts concerning zeros of symmetric solutions of some parabolic equations and some singular stationary solutions.

### 2.1 Zeros of radially symmetric solutions of parabolic equations

Consider radially symmetric solutions of semilinear parabolic equations.

$$\begin{cases}
  u_t = u_{rr} + \frac{n-1}{r} u_r + f(u) & \text{in } (0, 1) \times (0, T), \\
  \partial_r u > 0 & \text{on } \{1\} \times (0, T),
\end{cases}$$

where $f(\cdot)$ is in $C^1(\mathbb{R})$ and real analytic on $\mathbb{R}_+$ with $f(0) = 0$. It is well known that the number of zeros of a solution $u$ of (2.1) is finite for $t > 0$ (see [20], [2], [6]). We here note that the same is true for $u_r$. More precisely, we get the following result.

**Lemma 2.1.** Let $u(r,t)$ be a positive classical solution of (2.1). Then the number of zeros of $u_r(\cdot, t)$ in $(0, 1)$ is finite and nonincreasing for any $t \in (0, T)$.

**Proof.** We set $U(x,t) = u(|x|,t)$, Then $U(x,t)$ is a positive classical solution of $U_t = \Delta U + f(U)$ in $B_1 \times (0, T)$. From [13], $U(\cdot, t)$ is an analytic function in $B_1$ for any fixed $t \in (0, T)$. Suppose that there exists $t_1 \in (0, T)$ such that $u_r(\cdot, t_1)$ has infinitely many zeros in $[0, 1]$. Then there exist $r_\infty \in [0, 1]$ and $\{r_i\}_{i \in \mathbb{N}} \subset [0, 1]$ ($r_i \neq r_j$ if $i \neq j$) such that $u_r(r_i, t_1) = 0$ and $\lim_{i \to \infty} r_i = r_\infty$. Since $u_r(1, t_1) > 0$, it follows that $r_\infty \neq 1$. Hence, since $U(\cdot, t_1)$ is an analytic function in $B_1$, we obtain $\nabla U(\cdot, t_1) \equiv 0$ in $B_1$. However this contradicts that $\partial_r u(1, t_1) > 0$, which completes the proof. 

In order to investigate the zeros of \( u_r(\cdot, t) \) more precisely, we differentiate (2.1) with respect to \( r \) and set
\[
w(r, t) = u_r(r, t).
\]
Then \( w(r, t) \) satisfies the following linear equation.
\[
\begin{cases}
  w_t = w_{rr} + \frac{n-1}{r} w_r - \frac{n-1}{r^2} w + f'(u) w & \text{in } (0, 1) \times (0, T), \\
  w > 0 & \text{on } \{1\} \times (0, T).
\end{cases}
\]
The third term on the right-hand side has a singular coefficient \( 1/r^2 \). However, since the sign of this coefficient is negative, the classical maximum principle still holds for this equation. Hence we can show the nonincreasing property for the number of zeros of \( w(\cdot, t) \) by quite the same argument as in the proof of Theorem 6.15 in [16], where linear parabolic equations with bounded coefficients are treated. Then, by setting
\[
N_t = \text{the set of zeros of } w(\cdot, t) = u_r(r, t) \text{ in } (0, 1),
\]
\[
\#N_t = \text{the number of zeros of } w(\cdot, t) = u_r(r, t) \text{ in } (0, 1),
\]
we can obtain the following result.

**Lemma 2.2.** Let \( u(r, t) \) be as in Lemma 2.1 and let \( \#N_0 < \infty \). Then \( \#N_t \) is nonincreasing in \( t \in [0, T) \).

Furthermore, by virtue of Lemma 2.2, we can obtain more minute informations on the behavior of the set of zeros \( N_t \).

**Lemma 2.3.** Let \( u(r, t) \) be as in Lemma 2.1 and set \( z(t) = \sup N_t \). Then the number of discontinuity points of \( z(t) \) on \( (0, T) \) is at most \( N_0 \). Moreover \( z(t) \) has both the right-side and the left-side limits for any \( \tau \in (0, T) \) and it holds that
\[
z(t + 0) = \lim_{t \to \tau + 0} z(t) \leq \lim_{t \to \tau - 0} z(t) = z(t - 0).
\]

**Proof.** We first note that the discontinuity point of \( z(t) \) appears at \( t = t_d \) only when the curve \( z(t) \) merges with its nearest zero curve of \( u_r \) at \( t = t_1 \leq t_d \) and \( z(t) \) evaporates at \( t = t_d \in [t_1, T) \), more precisely, \( u_r(z(t), t) = 0 \) for \( t \in [t_1, t_d] \) and \( u_r(z(t_d), t) \neq 0 \) for \( t \in (t_d, t_d + \delta) \) for some \( \delta > 0 \). Hence the number of discontinuous points are at most \( N_0 \). Now we claim that \( z(t) \) has the left-side limit for all \( t \in (0, T) \). Suppose that \( z(t) \) does not have the left-side limit at \( t_1 \in (0, T) \). Then it should hold that \( r_a := \liminf_{t \to t_1 - 0} z(t) < \limsup_{t \to t_1 - 0} z(t) =: r_b \). By the continuity of \( u_r(r, t) \), we find that \( u_r(r, t_1) = 0 \) for \( r \in (r_a, r_b) \). However this contradicts Lemma
2.2, whence the claim follows. In the same manner, we can verify that \( z(t) \) has the right-side limit for all \( t \in (0, T) \).

Since \( u_r(z(t), t) = 0 \) and \( u_r(r, t) > 0 \) for all \( r \in (z(t), 1) \), by letting \( t \to \tau \pm 0 \), we get \( u_r(z(\tau \pm 0), \tau) = 0 \). Furthermore, by the strong maximum principle and the boundary condition \( u_r(1, \tau) > 0 \), we obtain \( u_r(r, \tau) > 0 \) for \( r \in (z(\tau - 0), 1) \). Hence we obtain \( z(\tau + 0) \leq z(\tau - 0) \).

\[ \tag{2.2} \]

\[ \begin{align*}
\phi'' + \frac{n-1}{r} \phi' &= q \phi^{2q-1} & \text{in } (\rho, 1), \\
\phi(\rho) &= \infty, \quad \phi(1) = \infty.
\end{align*} \]

We first note that, by Theorem 2.4 in [5], there exists a unique positive solution of (2.2), which is denoted by \( \phi_\rho(r) \). If \( \phi_\rho(r) \) vanishes at \( r = r_0 \), then by (2.2), we get \( \phi''_\rho(r_0) > 0 \), i.e., \( \phi_\rho(r) \) attains its minimum at \( r = r_0 \). Hence we can conclude that \( \phi_\rho(r) \) has a unique minimum point, which we denote by \( \rho_1 \). Moreover we can obtain more information about the asymptotic behavior of solutions of (2.2) near the boundary as follows.

**Lemma 2.4.** For any \( \rho \in (0, 1) \) there exist \( \delta_\rho > 0 \) and \( A_\rho > 0 \) such that

\[ 1 - A_\rho (1 - r) < \phi_\rho(r)^{-q} \phi'_\rho(r) < 1, \quad r \in (1 - \delta_\rho, 1). \]

**Proof.** Multiplying (2.2) by \( \phi'_\rho(r) \) and integrating over \( (\rho_1, r) \), we get

\[ \phi'_\rho(r)^2 = \phi_\rho(r)^{2q} - \phi_\rho(\rho_1)^{2q} - \int_{\rho_1}^{r} \frac{2(n-1)}{s} \phi'_\rho(s)^2 \, ds. \]

Since \( \phi'_\rho(r) > 0 \) for \( r \in (\rho_1, 1) \), this implies \( \phi_\rho(r)^{-q} \phi'_\rho(r) < 1 \) for \( r \in (\rho_1, 1) \). Therefore the last term in (2.4) is estimated by

\[ \int_{\rho_1}^{r} \frac{2(n-1)}{s} \phi'_\rho(s)^2 \, ds \leq \frac{2(n-1)}{\rho_1} \int_{\rho_1}^{r} \phi'_\rho(s)^q \phi_\rho(s)^q \, ds \leq \frac{2(n-1)}{(q+1) \rho_1} \phi_\rho(r)^{q+1}. \]

Using the fact that \( \phi_\rho(\rho_1)^{2q} \leq \phi_\rho(r)^{q+1} \phi_\rho(\rho_1)^{q-1} \) for \( r \in (\rho_1, 1) \) and putting \( c_\rho = 1 + 2(n-1)/(q+1) \rho_1 \), we obtain

\[ \phi'_\rho(r)^2 \geq \phi_\rho(r)^{2q} - c_\rho \phi_\rho(r)^{q+1} = \phi_\rho(r)^{2q} \left( 1 - c_\rho \phi_\rho(r)^{-(q-1)} \right). \]

From \( \phi^{-q}_\rho(r) \phi'_\rho(r) \leq 1 \) for \( r \in (\rho_1, 1) \), we easily get
\[
\phi_\rho(r)^{-(q-1)} \leq (q - 1)(1 - r). \tag{2.5}
\]

Thus we deduce
\[
\phi_\rho^{-2q} \phi_\rho'(r)^2 \geq 1 - c_\rho (q - 1)(1 - r).
\]

We here take \( \delta_\rho \in (\rho, 1) \) such that \( c_\rho (q - 1)(1 - r) \leq 1/2 \) for \( r \in (1 - \delta_\rho, 1) \). Then, by Taylor’s expansion, there exists \( A_\rho \) such that \( \sqrt{1 - c_\rho (q - 1)(1 - r)} \geq 1 - A_\rho (1 - r) \) for \( r \in (1 - \delta_\rho, 1) \). Thus the proof is completed. \( \square \)

Throughout this paper, we denote by \( \phi_\rho(r) \) the unique solution of (2.2) and by \( \rho_1 \) its minimum point. Moreover we set
\[
\psi_\rho(r) = \phi_\rho(r)^{-(q-1)}.
\]

From above arguments, we can derive that there exist \( \kappa_\rho > 0 \) and \( \delta_\rho > 0 \) such that
\[
-\bar{q} \leq \psi_\rho'(r) \leq -\bar{q} + \frac{\kappa_\rho}{2} (1 - r), \quad r \in (1 - \delta_\rho, 1),
\]
\[
\bar{q} (1 - r) - \kappa_\rho (1 - r)^2 \leq \psi_\rho(r) \leq \bar{q} (1 - r), \quad r \in (1 - \delta_\rho, 1).
\tag{2.6}
\]

In fact, the first estimate and the second inequality of the second estimate are nothing but (2.3) and (2.5) and the first inequality of the second estimate can be derived from the integration of the second inequality of the first estimate over \( (r, 1) \).

### 3 Finite time blow-up

In this section, we study the finite time blow-up phenomena of positive solutions of (1.1) for the critical case \( p = 2q - 1, a = q \). Our method is based on a comparison argument. To construct a suitable comparison function, we first consider the case where \( \Omega \) is the unit ball \( B_1 \) and the initial data is radially symmetric and denote \( Q_0^R \) by \( Q_T \). We study the behaviour of solutions in terms of a new unknown function \( v \) introduced in Introduction:
\[
v(r,t) = u(r,t)^{-(q-1)}. \tag{3.1}
\]

It is easy to see that \( v(r,t) \) satisfies (1.2). To show a finite time blow-up of \( u(r,t) \), it is sufficient to show that \( v(r,t) \) vanishes in a finite time. First we show that the vanishing can occur only on the boundary.

**Lemma 3.1.** There exists a number \( \rho \in (0, 1) \) depending on \( v_0(r) \) such that the positive solution \( v(r,t) \) of (1.2) satisfies
\[
v(r,t) \geq \begin{cases} 
\psi_\rho(\rho_1) & \text{if } r \in (0, \rho_1), \\
\psi_\rho(r) & \text{if } r \in (\rho_1, 1).
\end{cases}
\]
Proof. We set \( \hat{v}_\rho(r) = \psi_\rho(r) \) if \( r \in (\rho_1, 1) \) and \( \hat{v}_\rho(r) = \psi_\rho(r) \) if \( r \in (0, \rho_1] \). Then, since \( \hat{v}_\rho''(r) = 0 > \hat{v}_\rho''(\rho_1 + 0) = -\left(\frac{q}{n} \rho_{\rho^{n-1}}^n\right) \) for \( r \in (0, \rho_1) \), it is easy to see that \( \hat{v}_\rho \in H_r^2(0, 1) \) and

\[
-\partial_{rr} \hat{v}_\rho - \frac{n-1}{r} \partial_r \hat{v}_\rho - \frac{\hat{q}}{\hat{v}} (\hat{q}^2 - (\partial_r \hat{v}_\rho)^2) \leq 0
\]

for a.e. \( r \in (0, 1) \). From (2.6), we note that \( 0 \leq \psi_\rho(r) \leq \hat{q} (1 - r) \) in \((\rho_1, 1)\). Hence there exists \( \rho \in (0, 1) \) such that \( v_0(r) \geq \hat{v}_\rho(r) \) for \( r \in (0, 1) \). Then, noting that \( \partial_r \hat{v}_\rho(1) = -\hat{q} \), by a standard comparison argument, we obtain \( v(r, t) \geq \hat{v}_\rho(r) \) in \((0, 1) \times (0, T)\), which completes the proof. \( \square \)

The following lemma plays a crucial role to show the finite time vanishing.

**Lemma 3.2.** Let \( v(r, t) \in C_{r,t}^{2,1}(\overline{Q_{T'}}) \) be a positive solution of (1.2) for some \( T' > 0 \). If \( \partial_r v_0 \geq -\hat{q} \), then it holds that \( v(r, t) \geq -\hat{q} \) in \( Q_{T'} \).

Proof. Differentiate (1.2) with respect to \( r \) and set \( \xi(r, t) = v(r, t) + \hat{q} \), then we formally get

\[
\xi_t = \xi_{rr} + \frac{n-1}{r} \xi_r - \frac{n-1}{r^2} \xi - \frac{\hat{q}v_r (\hat{q} - v_r)}{v^2} \xi - \frac{2\hat{q}v_r}{v} \xi_r + \frac{(n-1)\hat{q}}{r^2} \xi \quad \text{in} \quad Q_{T'}.
\]

Since all terms except \( \xi_{rr} \) appearing in this equality exist in the classical sense, by the regularity assumption on \( v \), we can easily see that \( \xi_{rr} \) also exists and this equation is satisfied in the classical sense. Furthermore, by assumption on \( v_0 \), we find that \( \xi \geq 0 \) so \( \xi_- = 0 \) on \( \{1\} \times (0, T') \cup (0, 1) \times \{0\} \). Then, multiplying this equation by \( \xi_- r^{n-1} \) and integrating over \((0, 1)\), we obtain

\[
\frac{1}{2} \frac{d}{dt} \|\xi_-\|^2 + \|\partial_r \xi_-\|^2 \leq \int_0^R \left( \frac{\hat{q}}{v^2} |v_r (\hat{q} - v_r)| \xi_- + \frac{2\hat{q}v_r}{v} |\partial_r \xi_-|\xi_- \right) r^{n-1} dr,
\]

where we used the fact that \( 2\xi_\xi = -\partial_t \xi_-^2 \) and \( \xi_- = -\xi_-^2 \leq 0 \). Since \( v(r, t) \) is positive and \( v_r(r, t) \) is uniformly bounded in \( Q_{T'} \), applying Schwarz’s inequality for the second term of the right-hand side, by Gronwall’s inequality, we can deduce that \( \xi_- \equiv 0 \), whence it follows \( v_r \geq -\hat{q} \) in \( Q_{T'} \), which completes the proof. \( \square \)

**Lemma 3.3.** Let \( v_0(r) \) be a smooth positive function with \( \partial_r v_0(1) = -\hat{q} \) and \( \partial_r v_0(r) \geq -\hat{q} \) for \( r \in (0, 1) \), and let \( v(r, t) \) be the corresponding solution of (1.2) satisfying the same regularity as in Lemma 3.2. Then \( v(r, t) \) vanishes on the boundary in a finite time.
Here by Lemma 3.3, we note that \( w_u \geq -\bar{q} \) for \( t \in (0, T) \). As a consequence, according to a boundary condition \( \partial_r v(1, t) = -\bar{q} \), it is verified that \( \partial_r v(1, t) \leq 0 \). Hence, by using (1.2), we can derive

\[
v_t(1, t) \leq -(n-1)\bar{q}, \quad t \in (0, T).
\]

Hence \( v(1, t) \) vanishes in a finite time. Furthermore, by Lemma 3.1, the vanishing occurs only on the boundary.

**Proof of Theorem 1.1.** Now we consider the general domain case. Let \( u(x, t) \) be a positive solution of (1.1). We set \( u_R(x, t) = R^{1/(q-1)} u(R x, R^2 t) \) with \( R = \sup_{x \in \Omega} \text{dist}(x, 0) \). Then \( u_R(x, t) \) gives a solution of (1.1) with \( \Omega \) and \( u_0 \) replaced by \( \Omega_R = \{ x \in \mathbb{R}^n ; R x \in \Omega \} \) and \( u_{R0}(x) = R^{1/(q-1)} u_0(R^2 x) \). Here we choose a positive smooth function \( w_0 \) satisfying \( w_0(|x|) \leq u_{R0}(x) \) in \( \Omega_R \), \( \partial_r w_0(1) = w_0(1)^q \) and \( 0 \leq \partial_r w_0(r) \leq w_0(r)^q \) for \( r \in (0, 1) \). For example, if we set \( w_0(r) = w_1(r) + \alpha(1-r)^2 \), \( w_1(r) = (q-1)^{1/(1-q)}(R_1 - r)^{1/(q-1)} \) and \( 2\alpha = (q-1)^{q/(1-q)}R_1^{q/(1-q)} \), then \( w_0(r) \) satisfies the desired properties for a suitable choice of \( R_1 > 1 \). In fact, since \( \partial_r w_0(1) = \partial_r w_1(1) - 2\alpha(1-r) \) for \( r \in [0, 1] \) and \( \partial_r w_1(r) = w_1(r)^q \) for \( r \in [0, 1] \), it is easy to check that \( \partial_r w_0(1) = w_0(1)^q \), \( 0 \leq \partial_r w_0(r) \leq w_0(r)^q \) and \( \partial_r w_0(0) = 0 \).

Furthermore, by choosing \( R_1 > 1 \) sufficiently large, we can check the condition \( w_0(|x|) \leq u_{R0}(x) \).

We denote by \( w(|x|, t) \) the solution of (1.1) with the initial data \( w_0(|x|) \) and with \( \Omega \) replaced by \( B_1 \). We set \( z(r, t) = w(r, t)^{-\bar{q}} \). Since \( 0 \leq \partial_r w \leq w^q \) is equivalent to \( -\bar{q} \leq \partial_r z \leq 0 \), by Lemma 3.2, it follows that \( \partial_r w(|x|, t) \leq w(|x|, t)^q \) for \( (x, t) \in B_1 \times (0, T) \). Hence we obtain

\[
\partial_r w(|x|, t) \leq w(|x|, t)^q \quad \text{on} \quad \partial \Omega_R \times (0, T).
\]

Thus \( w(|x|, t) \) turn out to be a sub-solution of (1.1) in \( \Omega_R \), so by the comparison theorem, we obtain

\[
u(x, t) \geq w(|x|, t), \quad (x, t) \in \Omega_R \times (0, T).
\]

Here by Lemma 3.3, we note that \( w(|x|, t) \) blows up on \( \partial B_1 \) in a finite time. Hence \( u(x, t) \) also blows up in a finite time. For the case where \( \Omega \) is a ball, from Lemma 3.1, solutions blow up only on the boundary. Thus the proof is completed.

### 4 Existence of a blow-up profile

In this section, we study the asymptotic behavior of solutions of (1.1) near the blow-up time. We here restrict ourselves to the positive radially symmetric solutions in
the unit ball. In the polar coordinates, (1.1) can be rewritten as

\[
\begin{cases}
    u_t = u_{rr} + \frac{n-1}{r}u_r - qu^{2q-1} & \text{in } Q_T, \\
    \partial_r u = u^q & \text{on } \{1\} \times (0, T), \\
    u(r, 0) = u_0(r) & \text{in } (0, 1),
\end{cases}
\]

where \(Q_T = (0, 1) \times (0, T)\). We are going to analyse the behavior of solutions \(u(r, t)\) in terms of \(v(r, t) = u(r, t)^{-(q-1)}\). Our goal of this section is to give a proof for Theorem 1.2, which assures the existence of the blow-up (vanishing) profile.

In this section, \(u(r, t)\) is always assumed to be a classical solution of (4.1), \(v(r, t)\) is defined by (3.1) and \(T > 0\) denotes the blow-up time of \(u(r, t)\) (the vanishing time of \(v(r, t)\)). From Lemma 2.1, without loss of generality, we can assume that

the number of zeros of \(u_r(\cdot, t)\) on \([0, 1]\) is finite for \(t \in [0, T]\). \((a1)\)

Moreover, by virtue of Theorem 1.1, (2.6), Lemma 3.1 and the parabolic regularity theory, we can assume that for any \(R \in (0, 1)\) there exists \(C_R > 0\) such that

\[
u(r, t) + |u_t(r, t)| + |u_r(r, t)| + |u_{rr}(r, t)| \leq C_R \quad \text{in } (0, R) \times (0, T),
\]

and there exist \(\rho \in (0, 1)\) and \(\kappa > 0\) such that

\[
v(r, t) \geq \psi_\rho(r) \geq \bar{q}(1 - r) - \kappa (1 - r)^2 \quad \text{in } (1 - \delta_\rho, 1) \times (0, T),
\]

which are assumed throughout this section.

We can also assume that for \(\varepsilon > 0\) there exists \(c_\varepsilon > 0\) such that

\[
u(r, t) \geq c_\varepsilon \quad \text{in } (0, 1) \times (\varepsilon, T).
\]

\((c)\)

In fact, let \(u_1(r, t)\) be a unique solution of

\[
\begin{cases}
    u_t = u_{rr} + \frac{n-1}{r}u_r - qu^{2q-1} & \text{in } Q_T, \\
    \partial_r u = 0 & \text{on } \{1\} \times (0, T), \\
    u(r, 0) = u_0(r) & \text{in } (0, 1),
\end{cases}
\]

Then, by a strong maximum principle, for \(\varepsilon > 0\) there exists \(c_\varepsilon > 0\) such that

\[
u_1(r, t) \geq c_\varepsilon \quad \text{in } (0, 1) \times (\varepsilon, T).
\]

\((c)\)

By a comparison argument, we note that \(u(r, t) \geq u_1(r, t)\), which implies (c).
4.1 Estimate for $v_r$

We recall that the nearest zero of $u_r(\cdot, t)$ to the boundary $\{r = 1\}$ is defined by

$$z(t) = \sup \{ r \in [0, 1) ; u_r(r, t) = 0 \} = \sup \mathcal{N}_t.$$\hspace{1cm} (4.1)

**Lemma 4.1.** Let $z_0 = \lim_{t \to 0^+} z(t)$. Then it holds that

$$\max_{r \in [0, z(t)]} |u_r(r, t)| \leq \max_{r \in [0, z_0]} |\partial_r u_0(r)|, \quad t \in (0, T)$$\hspace{1cm} (4.2)

as long as $\mathcal{N}_t$ is not empty.

**Proof.** We assume that $\mathcal{N}_t$ is not empty for any $t \in (0, T')$ with some $T' \in (0, T]$. By Lemma 2.3, the number of discontinuous points of $z(t)$ is finite. We denote these points by $0 < t_1 < t_2 < \cdots < t_\ell < T'$, and set $t_0 = 0$, $t_{\ell+1} = T'$. Since $z(t)$ is continuous on $\bigcup_{i=0}^\ell (t_i, t_{i+1})$, the set $O_i = \{(r, t) ; 0 < r < z(t), t \in (t_i, t_{i+1})\}$ becomes an open connected set in $Q_T$ for $i = 0, \cdots, \ell$. To estimate $u_r(r, t)$, differentiating (4.1) with respect to $r$ and setting $w(r, t) = u_r(r, t)$, we get

$$w_t = w_{rr} + \frac{n-1}{r} w_r - \frac{n-1}{r^2} w - q (2q-1) u^{2q-2} w.$$\hspace{1cm} (4.3)

We set $M_0 = \max_{r \in [0, z_0]} |w(r, 0)|$. Suppose that there exists a point $(r_0, \tau_0) \in O_0$ such that $w(r_0, \tau_0) > M_0$. Then there exists another point $(r_1, \tau_1) \in O_0$ with $\tau_1 < \tau_0$ such that $w_t(r_1, \tau_1) \geq 0$, $w_r(r_1, \tau_1) = 0$, $w_{rr}(r_1, \tau_1) \leq 0$ and $w(r_1, \tau_1) \geq M_0$. However this contradicts (4.3). Hence it is verified that $\sup_{O_0} w(r, t) \leq M_0$. By the same manner, we can verify that $\inf_{O_0} w(r, t) \geq -M_0$. As a consequence, we obtain

$$\sup_{O_0} |w(r, t)| \leq M_0.$$\hspace{1cm}

By Lemma 2.3, we note that $z(t)$ has the left-hand limit for $t \in (0, T')$. Hence, by letting $t \to t_1 - 0$, we have

$$\max_{0 \leq r \leq z(t_1-0)} |w(r, t_1)| \leq M_0.$$\hspace{1cm} (4.4)

Moreover, recalling that Lemma 2.3 assures that $z(t_1 + 0) \leq z(t_1 - 0)$ and repeating the same procedure as above a finite time, we obtain

$$\sup_{O_i} |w(r, t)| \leq M_0, \quad \max_{0 \leq r \leq z(t_i-0)} |w(r, t_i)| \leq M_0 \quad \text{for } i \in \{0, \cdots, \ell\},$$

which implies (4.2). \hfill $\square$
Lemma 4.2. There exist \( r_0 \in (0, 1) \) and \( t_0 \in (0, T) \) such that \( u_r(r, t) > 0 \) for \( (r, t) \in (r_0, 1) \times (t_0, T) \).

Proof. From Lemma 2.2, the number of zeros of \( u_r(\cdot, t) \) is nonincreasing. Hence if \( u_r(\cdot, T') > 0 \) for some \( T' \in (0, T) \), the assertion of this lemma is obvious. Hence it suffices to consider the case where \( u_r(\cdot, t) \) has zeros for \( t \in (0, T) \). From Lemma 4.1, we note that for \( r \in (0, z(t)) \)

\[
  u(r, t) = u(0, t) + \int_0^r u_r(s, t) ds \leq u(0, t) + \max_{r \in [0, z_0]} |\partial_r u_0(r)|
\]

\[
  \leq \sup_{t \in (0, T)} u(0, t) + \max_{r \in [0, z_0]} |\partial_r u_0(r)| =: M. \tag{4.4}
\]

From (b), it is clear that \( M < \infty \). Applying Proposition 4.10, which will be given later, with \( \alpha = \max(\alpha_1, 3M) \), we obtain \( u(r, t) \geq \chi_\alpha(r) \) for \( (r, t) \in (\zeta_\alpha, 1) \times (t_\alpha, T) \).

Then there exists \( r_0 \in (\zeta_\alpha, 1) \) such that \( \chi_\alpha(r) \geq 2M \) for \( r \in (r_0, 1) \). Hence, from (4.4), we obtain \( z(t) < r_0 \) for \( t \in (t_\alpha, T) \), which completes the proof. \( \square \)

By virtue of Lemma 4.2, we can assume without loss of generality that

\[
  u_r(r, t) > 0 \quad \text{in} \quad (r_0, 1) \times [0, T). \tag{a2}
\]

For the rest of this section, we also assume (a2). Next we give estimates for \( v_r(r, t) \).

Differentiating (1.2) with respect to \( r \) and set \( V(r, t) = v_r(r, t) \), then we obtain

\[
  \begin{cases}
    V_t = V_{rr} + \frac{n-1}{r} V_r - \frac{n-1}{r^2} V - \frac{\hat{q}}{v^2} V (\hat{q}^2 - V^2) - \frac{2\hat{q}}{v} V V_r & \text{in} \ Q_T, \\
    V = -\bar{q} & \text{on} \ \{1\} \times (0, T).
  \end{cases} \tag{4.5}
\]

We define a operator \( L_r \) associated with (4.5) by

\[
  L_r V = V_t - V_{rr} - \frac{n-1}{r} V_r + \frac{n-1}{r^2} V + \frac{\hat{q}}{v^2} V (\hat{q}^2 - V^2) + \frac{2\hat{q}}{v} V V_r.
\]

Lemma 4.3. There exists \( b > 0 \) such that

\[
  \begin{align*}
    v_r(r, t) & \leq -\bar{q} + b(1-r) & \text{in} \ (1 - \bar{q}/b, 1) \times (0, T), \\
    v_t(r, t) & \geq -(b + nq) & \text{on} \ \{1\} \times (0, T).
  \end{align*}
\]

Proof. We set \( \bar{V}(r) = -\bar{q} + b(1-r) \) and \( r_1(b) = 1 - \bar{q}/b \), where \( b \) is a positive constant to be fixed later. Then it is easily seen that \( \bar{V}(r) < 0 \) for \( r \in (r_1, 1) \). Hence, from (b), we verify that for \( r \in (\max\{1 - \delta_\rho, r_1\}, 1) \)

\[
  L_r \bar{V} \geq \frac{(n-1)b}{r} - \frac{(n-1)\bar{q}}{r^2} + \frac{\hat{q} |\bar{V}|}{v^2} (\bar{V}^2 - \bar{q}^2 + 2bv)
  \geq \frac{(n-1)b}{r} - \frac{(n-1)\bar{q}}{r^2} + \frac{\hat{q} |\bar{V}|}{v^2} (b^2 - 2kb) (1-r)^2.
\]
Now we choose $b > 0$ large enough such that $r_1(b) \geq \max\{r_0, 1 - \delta_p\}$ and
\[ L_r \bar{V} \geq 0 \quad \text{in } (r_1, 1), \quad \bar{V} \geq \partial_r v_0 \quad \text{in } (r_1, 1). \]
By the boundary condition, it is clear that
\[ \bar{V} = v_r = -\bar{q} \quad \text{on } \{1\} \times (0, T). \]
Moreover, from $\bar{V}(r_1) = 0$ and (a2), we see that
\[ \bar{V} \geq v_r \quad \text{on } \{r_1\} \times (0, T). \]
Hence, applying the comparison argument in $(r_1, 1) \times (0, T)$, we conclude that $v_r(r, t) \leq -\bar{q} + b(1 - r)$ in $(r_1, 1) \times (0, T)$. Consequently, it follows from $v_r(1, t) = -\bar{q}$ that $v_{rr}(1, t) \geq -b$ for $t \in (0, T)$. Since $v(r, t)$ is a solution of (1.2), we obtain
\[ v_t(1, t) = v_{rr}(1, t) - (n - 1)\bar{q} \geq -b - nq, \]
which completes the proof. \hfill \Box

### 4.2 Estimate for $(v_r^2 - \bar{q}^2)/v$

In this subsection, we choose a constant $b > 0$ given in Lemma 4.3 such that $b > \kappa$, where $\kappa$ is a positive constant in (b). For simplicity of notations, we set
\[ \bar{r} = 1 - \bar{q}/2b, \quad Q = (\bar{r}, 1) \times (0, T). \]
We denote by $\partial_p Q$ a parabolic boundary of $Q$, which is given by
\[ \partial_p Q = \{\bar{r}, 1\} \times (0, T) \cup (\bar{r}, 1) \times \{0\}. \]

**Lemma 4.4.** There exist $c_1 > 0$ and $\gamma > 0$ such that
\[ u_t(r, t) \leq c_1 e^{\gamma t} u_r(r, t) \quad \text{in } Q. \]

**Proof.** From Lemma 4.3, there exists $c_0 > 0$ such that $-v_t(1, t) \leq c_0 \bar{q} = -c_0 v_r(1, t)$ for $t \in (0, T)$. Hence, by $v(r, t) = u(r, t)^{q-1}$, we see that
\[ u_t(1, t) \leq c_0 u_r(1, t), \quad t \in (0, T). \]
Furthermore, from Lemma 4.3, we have $v_r(\bar{r}, t) \leq -\bar{q}/2$ for $t \in (0, T)$, which implies that $u_r(\bar{r}, t) \geq u(\bar{r}, t)^q/2$ for $t \in (0, T)$. By the strong maximum principle, we note
that \( A_0 := \inf_{t \in (0,T)} u(\bar{r}, t) > 0 \). Hence we get \( \inf_{t \in (0,T)} u_r(\bar{r}, t) \geq A_0^q/2 \). From (b), we see that \( A := \sup_{t \in (0,T)} u_t(\bar{r}, t) < \infty \). Therefore we get
\[
 u_t(\bar{r}, t) \leq \left( \frac{2A}{A_0^q} \right) u_r(\bar{r}, t), \quad t \in (0, T).
\]
By the regularity assumption of solutions, there exists \( c_1 > 0 \) such that
\[
 \partial_t u_0(r, 0) \leq c_1 \partial_r u_0(r, 0), \quad r \in (\bar{r}, 1).
\]
By virtue of these inequalities, there exists \( c > 0 \) such that
\[
 u_t(r, t) \leq cu_r(r, t) \quad \text{on } \partial Q.
\] (4.6)
We set \( W = u_t \) and \( w = ce^{\gamma t}u_r \), where \( \gamma = (n-1)/\bar{r}^2 \). Then we get
\[
 \partial_t W = \partial_{rr} W + \frac{n-1}{r} \partial_r W - q(2q-1)u^{2q-2}W \quad \text{in } Q.
\] (4.7)
Moreover, from \( w \geq 0 \) in \( Q \), we see that
\[
 \partial_t w = \partial_{rr} w + \frac{n-1}{r} \partial_r w + \left( \gamma - \frac{n-1}{r^2} \right) w - q(2q-1)u^{2q-2}w \\
\geq \partial_{rr} w + \frac{n-1}{r} \partial_r w - q(2q-1)u^{2q-2}w \quad \text{in } Q.
\]
Since \( u_r \geq 0 \) in \( Q \), it is clear that \( cu_r \leq w \) in \( Q \). Hence (4.6) implies that
\[
 W(r, t) \leq w(r, t) \quad \text{on } \partial_p Q.
\]
Applying the comparison argument, we obtain \( W(r, t) \leq w(r, t) \) in \( Q \), which completes the proof. \( \square \)

**Lemma 4.5.** There exists \( c_2 > 0 \) such that
\[
 v_r(r, t) \geq -c_2 e^{\gamma t} \quad \text{in } Q.
\]

**Proof.** From Lemma 4.3, we note that \( v_r^2 - q^2 \geq -2\bar{q}b(1-r) \) in \( Q \). Hence, from (b), it is verified that
\[
 \frac{v_r^2 - q^2}{v} \geq \frac{-2\bar{q}b(1-r)}{v} \geq \frac{-2\bar{q}b(1-r)}{\bar{q}(1-r) - \kappa(1-r)^2}.
\]
By the definition of \( \bar{r} \) and \( b \geq \kappa \), we see that
\[
 \bar{q} - \kappa(1-r) \geq \bar{q} - \kappa(1-\bar{r}) = \bar{q} - \kappa\bar{q}/2b \geq \bar{q}/2, \quad r \in (\bar{r}, 1).
\]
This implies that
\[
\frac{v_r^2 - \bar{q}^2}{v} \geq -4b \quad \text{in } Q.
\] (4.8)

Hence we obtain
\[
v_{rr} = v_t - \frac{n - 1}{r} v_r + \frac{\hat{q}}{v} \left( v_r^2 - \bar{q}^2 \right) \geq v_t - 4\hat{q}b \quad \text{in } Q.
\]

As a consequence, it holds that
\[
[v_{rr}]_- \leq [v_t]_- + 4\hat{q}b \quad \text{in } Q.
\]

By Lemma 4.4, we note that $[v_t]_- \leq c_1 e^{\gamma t} |v_r|$ in $Q$. Hence we get
\[
\int_{\bar{r}}^1 [v_{rr}]_- \leq \int_{\bar{r}}^1 (c_1 e^{\gamma t} |v_r| + 4\hat{q}b) = \int_{\bar{r}}^1 (-c_1 e^{\gamma t} v_r + 4\hat{q}b) = -c_1 e^{\gamma} (v(1,t) - v(\bar{r},t)) + 4\hat{q}b(1 - \bar{r}) \leq c_1 e^{\gamma} v(\bar{r},t) + 4\hat{q}b.
\]

Therefore, from $v_{rr} = [v_{rr}]_+ - [v_{rr}]_-$, we obtain
\[
\int_{\bar{r}}^1 [v_{rr}]_+ = \int_{\bar{r}}^1 v_{rr} + \int_{\bar{r}}^1 [v_{rr}]_- = v_r(1,t) - v_r(\bar{r},t) + \int_{\bar{r}}^1 [v_{rr}]_- \leq |v_r(\bar{r},t)| + c_1 e^{\gamma} v(\bar{r},t) + 4\hat{q}b.
\]

This implies that
\[
\int_{\bar{r}}^1 |v_{rr}(r,t)| \leq |v_r(\bar{r},t)| + 2c_1 e^{\gamma t} v(\bar{r},t) + 8\hat{q}b.
\]

Thus we conclude that
\[
v_r(r,t) = v_r(1,t) - \int_{\bar{r}}^1 v_{rr}(\rho,t) d\rho \geq -\bar{q} - (|v_r(\bar{r},t)| + 2c_1 e^{\gamma} v(\bar{r},t) + 8\hat{q}b).
\]

Therefore, from (b), we complete the proof. \qed

**Lemma 4.6.** There exists $c_3 > 0$ such that
\[
v_r(r,t)^2 - \bar{q}^2 \leq c_3 e^{2\gamma t} v(r,t) \quad \text{in } Q.
\]
Proof. Multiplying (4.1) by $u_r$ and integrating over $(\bar{r}, r)$, we obtain
\[ u_r(r, t)^2 - u(r, t)^{2q} \leq u_r(\bar{r}, t)^2 + 2 \int_{\bar{r}}^{r} u_t u_r. \]
Since $u_r \geq 0$ in $Q$, by Lemma 4.4 and Lemma 4.5, it holds that
\[ u_r(r, t)^2 - u(r, t)^{2q} \leq u_r(\bar{r}, t)^2 + 2 c_1 c_2 e^{2\gamma t} u(r, t)^{q+1}, \quad (r, t) \in Q. \]
Hence we conclude that
\[ \left( \frac{v_r(r, t)}{\bar{q}} \right)^2 - 1 \leq \left( \frac{u_r(\bar{r}, t)^2}{u(r, t)^{q+1}} + \frac{2 c_1 c_2 e^{2\gamma t}}{\bar{q}(q + 1)} \right) v(r, t), \quad (r, t) \in Q, \]
which together with estimate (c) completes the proof. \hfill \Box

Lemma 4.7. $v_t(r, t)$ and $v_{rr}(r, t)$ are uniformly bounded on $Q$.

Proof. Note that $W(r, t) = u_t(r, t)$ is a solution of (4.7). Moreover, from Proposition 4.10, there exists $t_0 \in (0, T)$ such that $W(1, t) = u_t(1, t) \geq 0$ for $t \in (t_0, T)$. Hence we find that $W_1(r, t) = -\|[u_t]_-(t_0)\|_{L^\infty(0, 1)}$ becomes a sub-solution of (4.7) on $[t_0, T)$. Therefore, by the comparison argument, it is verified that
\[ u_t(r, t) = W(r, t) \geq -\|[u_t]_-(t_0)\|_{L^\infty(0, 1)} \quad \text{in } (0, 1) \times (t_0, T), \]
which implies that $\|[u_t]_-(t)\|_{L^\infty(0, 1)} \leq \|[u_t]_-(t_0)\|_{L^\infty(0, 1)}$ for $t \in (t_0, T)$. Hence, by (c), $[v_t]_+$ is uniformly bounded on $Q$. By virtue of Lemma 4.3 and Lemma 4.5, $|v_r|$ is uniformly bounded on $Q$. Hence a boundedness of $[v_t]_-$ follows from Lemma 4.4 and (c). Thus $v_t(r, t)$ is uniformly bounded on $Q$. Next we show a boundedness of $v_{rr}(r, t)$. From (4.8), there exists $c > 0$ such that
\[ \bar{q}^2 - v_r(r, t)^2 \leq cv(r, t) \quad \text{in } Q. \]
Hence, applying Lemma 4.6, we obtain
\[ |v_r(r, t)^2 - \bar{q}^2| \leq c' v(r, t) \quad \text{in } Q \]
for some $c' > 0$. Therefore a boundedness of $v_t, v_r, (v_r^2 - \bar{q}^2)/v$ in $Q$ implies a boundedness of $v_{rr}$ in $Q$. Thus the proof is completed. \hfill \Box
Proof of Theorem 1.2. From Lemma 4.7, (b) and (c), we find that $v, v_r, v_{rr}, v_t$ are uniformly bounded on $(0, 1) \times (0, T)$. Hence $v(r, t)$ converges to some function $\bar{v}_0(r) \in C([0, 1])$ uniformly on $[0, 1]$ as $t \to T - 0$. Furthermore we see that $\bar{v}_0 \in W^{2, \infty}_r(0, 1)$ and

$$v \in C([0, 1] \times [0, T]) \cap C^{2, 1}([0, 1] \times [0, T]), \quad v_t, v_{rr} \in L^\infty((0, 1) \times (0, T)).$$

Now we claim that $v_r \in C([0, 1] \times [0, T])$. From the boundary condition, we note that

$$|\bar{q} + v_r(r, t)| \leq \int_r^1 |v_{rr}(r, t)| \, ds \leq c(1 - r), \quad t \in (0, T). \quad (4.9)$$

Since $v_r \in C([0, 1] \times [0, T])$, from (4.9), we obtain $v_r \in C([0, 1] \times [0, T])$. Hence the claim is proved. Next we derive the blow-up rate estimate. From (b), (a2) and Theorem 1.1, we note that

$$\|u(t)\|_\infty = u(1, t), \quad t \sim T. \quad (4.10)$$

Moreover, from Lemma 4.7, there exists $c > 0$ such that $v_t(1, t) \geq c r$. Hence, integrating both sides on $(t, T)$, we obtain $v(1, t) \leq c(T - t)$. Since $v(r, t) = u(r, t)^{(q-1)}$, it follows that $u(1, t) \geq c^{-1/(q-1)}(T - t)^{-1/(q-1)}$. Therefore, from (4.10), we obtain (1.3). On the other hand, under the conditions: $u'_0(r) \leq u_0(r)^q$ in $(0, 1)$ and $u'_0(1) = u_0(1)^q$, we obtain from (3.2)

$$v_t(1, t) \leq -(n - 1)\bar{q}.$$ 

Hence, by the same way as above, we conclude (1.4). Thus the proof is completed.

To obtain a blow-up profile of the original solution $u(r, t)$, we set

$$\bar{u}_0(r) = \bar{v}_0(r)^{-1/(q-1)}.$$ 

Then, from Theorem 1.2, it is clear that $u(r, t)$ converges to $\bar{u}_0(r)$ in $C^2([0, 1 - \varepsilon])$ for any $\varepsilon > 0$ as $t \to T$. By the same way as in the proof of Lemma 4.1, we obtain the following lemma immediately.

Lemma 4.8. The number of zeros of $\partial_r \bar{u}_0$ in $(0, 1)$ is less than that of $\partial_r u_0$. We denote by $z_T = \lim_{t \to T} z(t) \in (0, 1)$ the nearest zero of $\partial_r \bar{u}_0$ to the boundary $\{r = 1\}$. Then if $\partial_r \bar{u}_0(r)$ has zeros in $(0, 1)$, it holds that

$$\max_{r \in [0, z_T]} |\partial_r \bar{u}_0(r)| \leq \max_{r \in [0, z_0]} |\partial_r u_0(r)|.$$
4.3 Monotonicity of solutions near blow-up time

Here we show the monotonicity of solutions near the blow-up time. Our argument is based on the intersection comparison argument given in [14]. For the intersection comparison argument, first we study the following ODE problem.

\[ \chi'' + \frac{n-1}{r}\chi' = q\chi^{2q-1}, \quad \chi(1) = \alpha, \quad \chi'(1) = \alpha^q \quad (\alpha > 0). \quad (4.11) \]

We denote by \( \chi_\alpha(r) \) the unique solution of (4.11). When \( \chi_\alpha(r) \) vanishes at some point \( r \in [0,1) \), we denote this point by \( \zeta_\alpha \). If \( \chi_\alpha(r) \) does not vanish in \((0,1)\), we define \( \zeta_\alpha = 0 \).

**Lemma 4.9.** There exists \( \alpha_0 > 0 \) such that \( \chi_\alpha(r) \) vanishes in \((0,1)\) for \( \alpha \geq \alpha_0 \). Moreover \( \chi_\alpha(r) \) is monotone increasing for \( r \in (\zeta_\alpha, 1) \) and satisfies

\[ \chi'_\alpha(r) \geq \alpha - \chi_\alpha(r), \quad r \in (\zeta_\alpha, 1). \quad (4.12) \]

**Proof.** Multiplying (4.11) by \( \chi'_\alpha \) and integrating over \((r,1)\), we obtain

\[ \chi'_\alpha(r)^2 = \chi_\alpha(r)^{2q} + \int_r^1 \frac{2(n-1)}{\rho} \chi'_\alpha(\rho)^2 d\rho, \quad (4.13) \]

whence it follows \( \chi'_\alpha(r) > 0 \). Hence, applying Schwarz’s inequality in (4.13) and using the fact that \( n \geq 2 \), we obtain

\[ \chi'_\alpha(r)^2 \geq \frac{2(n-1)}{(1-r)} \left( \int_r^1 \chi'_\alpha(\rho)d\rho \right)^2 \geq 2 (\alpha - \chi_\alpha(r))^2, \]

which implies (4.12). From (4.13), it follows that \( \chi'_\alpha \geq \chi^q_\alpha \) in \((\zeta_\alpha, 1)\). Hence we obtain

\[ \chi_\alpha(r) \leq (\bar{q}(1-r))^{-1/\bar{q}}, \quad r \in (\zeta_\alpha, 1). \quad (4.14) \]

Now we claim that \( \zeta_\alpha \in (0,1) \) if \( \alpha > 3\alpha_0 \) with \( \alpha_0 = (\bar{q}/2)^{-1/\bar{q}} \). Suppose that there exists \( \alpha > 3\alpha_0 \) such that \( \zeta_\alpha = 0 \). Then, from (4.14), we see that

\[ \chi_\alpha(r) \leq \alpha_0, \quad r \in (0,1/2]. \quad (4.15) \]

Hence, by (4.12), we get

\[ \chi'_\alpha(r) \geq \alpha - \alpha_0, \quad r \in (0,1/2]. \]

Integrating this inequality both sides over \((0,1/2)\), we obtain \( \chi_\alpha(1/2) \geq (\alpha - \alpha_0)/2 \). Hence, by (4.15), we conclude that \( \alpha \leq 3\alpha_0 \), which contradicts the assumption. Thus the proof is completed. \( \square \)
Proposition 4.10. Let $t_\alpha = \inf\{t \in (0, T); u(1, t) = \alpha\}$. Then there exists $\alpha_1 > 0$ depending on $u_0$ such that if $\alpha \geq \alpha_1$, then it holds that

$$u(r, t) > \chi_\alpha(r) \quad \text{in } (\zeta_\alpha, 1) \times (t_\alpha, T).$$

Moreover it holds that $u_t(1, t) \geq 0$ for $t \in (t_\alpha, T)$.

Proof. By Lemma 4.9, it is easy to see that there exists a sufficiently large $\alpha_1 > 0$ such that $\chi_\alpha$ intersects with $u_0$ exactly once in $(\zeta_\alpha, 1)$ for $\alpha \geq \alpha_1$. Since $u(r, t)$ blows up on the boundary $\{r = 1\}$, it holds that $\sup_{t \to T} u(1, t) = \infty$. Since the number of intersections of $\chi_\alpha$ and $u(\cdot, t)$ is nonincreasing (see Theorem 6.15 in [16]), by the definition of $t_\alpha$, it holds that

$$u(r, t_\alpha) > \chi_\alpha(r) \quad \text{in } (\zeta_\alpha, 1)$$

if $\alpha \geq \alpha_1$. Hence, by the comparison argument, we obtain

$$u(r, t) > \chi_\alpha(r) \quad \text{in } (\zeta_\alpha, 1) \times (t_\alpha, T)$$

if $\alpha \geq \alpha_1$.

Therefore for any $\alpha \geq \alpha_1$, it holds that $u(1, t_\alpha) = \alpha$ and $u(1, t) > \alpha = \chi_\alpha(1)$ for $t > t_\alpha$. This implies that $u_t(1, t_\alpha) \geq 0$ for $\alpha > \alpha_1$, which completes the proof. \qed

5 Existence of global solutions

In the previous section, we proved that a solution $v(r, t)$ can be continuously extended up to the vanishing time $T$. In this section, we construct a global solution $\bar{v}(r, t)$ beyond the vanishing time $T$. The goal of this section is to construct a global solution of

$$\begin{cases}
\xi_t = \xi_{rr} + \frac{n-1}{r} \xi_r + \frac{\bar{q}}{\xi} (\bar{q}^2 - \xi_r^2) & \text{in } (0, 1) \times (0, \infty), \\
\partial_r \xi = -\bar{q} & \text{on } \{1\} \times (0, \infty), \\
\xi(r, 0) = \bar{v}_0(r) & \text{in } (0, 1),
\end{cases}$$  \hspace{1cm} (5.1)

where $\bar{v}_0(r) = v(r, T)$. Once a solution $\xi(r, t)$ of (5.1) is constructed, then

$$\bar{v}(r, t) = \begin{cases}
v(r, t) & \text{if } t \in [0, T), \\
\bar{v}_0(r) & \text{if } t = T, \\
\xi(r, t - T) & \text{if } t \in (T, \infty)
\end{cases}$$ \hspace{1cm} (5.2)

gives the desired global solution.
Proposition 5.1. There exists a solution $\xi \in C([0,1] \times [0, \infty)) \cap C^{2,1}((0,1) \times (0, \infty))$ of (5.1) satisfying

(A) $\xi_r \in L^\infty((0,1) \times (0, \tau))$, $\xi_{rr}, \xi_t \in L^p_r((0,1) \times (0, \tau))$ for any $p \in (1, \infty)$, $\tau > 0$, 
(B) $\xi_r \in C([0,1] \times (0, \infty))$ and for any $\tau > 0$, there exists $c_\tau > 0$ such that 
$$|\xi_r(r,t) + \bar{q}| \leq c_\tau (1 - r) \quad \text{in} \ (0,1) \times (0, \tau),$$
(C) $\xi(1,t) = 0$ for $t \in (0, \infty)$.

Since the solution $\xi(r,t)$ given in Proposition 5.1 simultaneously satisfies two boundary conditions:
$$\xi(1,t) = 0 \quad \text{for} \ t \in (0, \infty), \quad \partial_r \xi(1,t) = -\bar{q} \quad \text{for} \ t \in (0, \infty),$$
we can not apply the standard parabolic theory. To construct solutions stated in Proposition 5.1, we introduce the following approximation problems.

\[
\begin{cases}
\xi_t = \xi_{rr} + \frac{n-1}{r} \xi_r + \frac{\bar{q}}{\bar{\xi}} (\bar{q}^2 - \xi_r^2) & \text{in} \ (0,1) \times (0, \infty), \\
\xi = \varepsilon & \text{on} \ \{1\} \times (0, \infty), \\
\xi(r,0) = \xi_{\varepsilon,0}(r) := \bar{v}_0(r) + \varepsilon & \text{in} \ (0,1).
\end{cases}
\tag{5.3}
\]

From Lemma 3.1 and lemmas given in Section 4, there exist $\bar{c} > 0$ and $\rho \in (0,1)$ independent of $\varepsilon > 0$ such that
$$\xi_{\varepsilon,0}(r) \geq \begin{cases}
\psi_\rho(r) + \varepsilon & \text{if} \ r \in (\rho_1, 1), \\
\psi_\rho(\rho_1) + \varepsilon & \text{if} \ r \in (0, \rho_1),
\end{cases}
\tag{b1}
$$
$$|\partial_r \xi_{\varepsilon,0}(r) + \bar{q}| \leq \bar{c}(1 - r), \quad r \in (0,1),$$
$$u_{\varepsilon,0}(r) + |\partial_r u_{\varepsilon,0}(r)| + |\partial_{rr} u_{\varepsilon,0}(r)| \leq \bar{C}_R, \quad r \in (0,R),$$
where $R \in (0,1)$ and $u_{\varepsilon,0}(r)$ is given by
$$u_{\varepsilon,0}(r) = \xi_{\varepsilon,0}(r)^{-1/(q-1)}.$$

Lemma 5.2. For any $\varepsilon > 0$, there exists a unique solution $\xi_{\varepsilon}(r,t)$ of (5.3) such that $\xi_{\varepsilon} \in C^{1,0}([0,1] \times [0, \infty)) \cap C^{2,1}([0,1] \times (0, \infty))$ and $\partial_t \xi_{\varepsilon} \in C([0, \infty); L^p_r((0,1)))$ for any $p \in [2, \infty)$.

Proof. To construct a solution of (5.3), we go back to the original equation.

\[
\begin{cases}
U_t = \Delta U - q U^{2q-1} & \text{in} \ B_1 \times (0, \infty), \\
U = \varepsilon^{-1/(q-1)} & \text{on} \ \partial B_1 \times (0, \infty), \\
U(x,0) = U_{\varepsilon,0}(x) = u_{\varepsilon,0}(|x|) & \text{in} \ B_1.
\end{cases}
\]
Since \( u_{\varepsilon,0}(|x|) \in W^{2,\infty}(B_1) \subset C(\overline{B_1}) \) and \( u_{\varepsilon,0}(1) = \varepsilon^{-1/(q-1)} \), there exists a unique solution \( U_{\varepsilon} \in C(\overline{B_1 \times [0, \infty)}) \cap C^{2,1}(\overline{B_1 \times (0, \infty)}) \). Set \( \nu_{\varepsilon}(x,t) = U_{\varepsilon}(x,t) - U_{\varepsilon,0}(x) \). Then \( \nu_{\varepsilon}(x,t) \) is a solution of

\[
\begin{aligned}
\partial_t \nu_{\varepsilon} &= \Delta \nu_{\varepsilon} + f_{\varepsilon}(x,t) \quad \text{in } B_1 \times (0, \infty), \\
\nu_{\varepsilon} &= 0 \quad \text{on } \partial B_1 \times (0, \infty) \cup B_1 \times \{0\},
\end{aligned}
\]

where \( f_{\varepsilon}(x,t) = \Delta U_{\varepsilon,0}(x) - q U_{\varepsilon}(x,t)^{2q-1} \). Since \( \nu_{\varepsilon}(\cdot,0) = 0 \) and \( f_{\varepsilon} \in L^{\infty}(B_1 \times (0, \tau)) \) for \( \tau > 0 \), the parabolic \( L^p \)-theory assures that \( \nabla \nu_{\varepsilon}, \partial_t \nu_{\varepsilon} \in L^p(B_1 \times (0, \tau)) \) for \( p \in (1, \infty) \) and \( \tau \in (0, \infty) \). Hence \( \nu_{\varepsilon} \) and \( U_{\varepsilon} \) are Hölder continuous on \( \overline{B_1 \times [0, \tau]} \) for \( \tau > 0 \). Let \( \mu_{\varepsilon}(x,t) \) be a unique solution of

\[
\begin{aligned}
\partial_t \mu_{\varepsilon} &= \Delta \mu_{\varepsilon} \quad \text{in } B_1 \times (0, \infty), \\
\mu_{\varepsilon} &= 0 \quad \text{on } \partial B_1 \times (0, \infty), \\
\mu_{\varepsilon}(x,0) &= \mu_{\varepsilon,0}(x) := U_{\varepsilon,0}(x) - \varepsilon^{-1/(q-1)} \quad \text{in } B_1.
\end{aligned}
\]

Since \( \mu_{\varepsilon,0} \in W^{2,\infty}(B_1) \) and \( \mu_{\varepsilon,0} = 0 \) on \( \partial B_1 \), it holds that \( \nabla \mu_{\varepsilon} \in C(\overline{B_1 \times [0, \infty)}) \) and \( \partial_t \mu_{\varepsilon} \in C([0, \infty); L^p(B_1)) \) for \( p \in [2, \infty) \). Moreover, since \( U_{\varepsilon}(x,t)^{2q-1} \) is Hölder continuous on \( \overline{B_1 \times [0, \tau]} \) for \( \tau > 0 \), there exists a unique solution \( \sigma_{\varepsilon}(x,t) \) of

\[
\begin{aligned}
\partial_t \sigma_{\varepsilon} &= \Delta \sigma_{\varepsilon} - q U_{\varepsilon}^{2q-1} \quad \text{in } B_1 \times (0, \infty), \\
\sigma_{\varepsilon} &= 0 \quad \text{on } \partial B_1 \times (0, \infty) \cup B_1 \times \{0\}
\end{aligned}
\]

such that \( \partial_t \sigma_{\varepsilon} \) and \( \nabla \sigma_{\varepsilon} \) are Hölder continuous on \( \overline{B_1 \times [0, \tau]} \) for \( \tau > 0 \). By the definition of \( \mu_{\varepsilon} \) and \( \sigma_{\varepsilon} \), it is verified that \( \hat{U}_{\varepsilon}(x,t) := \mu_{\varepsilon}(x,t) + \sigma_{\varepsilon}(x,t) + \varepsilon^{-1/(q-1)} \) is a solution of

\[
\begin{aligned}
\partial_t \hat{U}_{\varepsilon} &= \Delta \hat{U}_{\varepsilon} - q U_{\varepsilon}^{2q-1} \quad \text{in } B_1 \times (0, \infty), \\
\hat{U}_{\varepsilon} &= \varepsilon^{-1/(q-1)} \quad \text{on } \partial B_1 \times (0, \infty), \\
\hat{U}_{\varepsilon}(x,0) &= U_{\varepsilon,0}(x) \quad \text{in } B_1.
\end{aligned}
\]

Hence, by uniqueness, it holds that \( U(x,t) \equiv \hat{U}_{\varepsilon}(x,t) \). By the regularity of \( \hat{U}_{\varepsilon} \), we see that \( \nabla \hat{U}_{\varepsilon} \in C(\overline{B_1 \times [0, \infty)}) \), \( \partial_t \hat{U}_{\varepsilon} \in C([0, \infty); L^p(B_1)) \) for \( p \in [2, \infty) \). Finally we set \( u_{\varepsilon}(|x|,t) = \hat{U}_{\varepsilon}(x,t) \). Then \( \xi_{\varepsilon}(r,t) = u_{\varepsilon}(r,t)^{-1/(q-1)} \) is the desired solution. \( \square \)

For the rest of this section, \( \xi_{\varepsilon}(r,t) \) stands for the unique solution of (5.3) given in Lemma 5.2 and set \( u_{\varepsilon}(r,t) = \xi_{\varepsilon}(r,t)^{-1/(q-1)} \). Then \( u_{\varepsilon}(r,t) \) gives a solution of

\[
\begin{aligned}
\partial_t u_{\varepsilon} &= \partial_{rr} u_{\varepsilon} + \frac{n-1}{r} \partial_r u_{\varepsilon} - q u_{\varepsilon}^{2q-1} \quad \text{in } (0,1) \times (0, \infty), \\
u_{\varepsilon} &= \varepsilon^{-1/(q-1)} \quad \text{on } \{1\} \times (0, \infty), \\
u_{\varepsilon}(r,0) &= u_{\varepsilon,0}(r) \quad \text{in } (0,1).
\end{aligned}
\]
From (b1), the comparison argument (see proof of Lemma 3.1) shows that

\[ \xi_\varepsilon(r, t) \geq \begin{cases} 
\psi_\rho(\rho_1) & \text{if } r \in (0, \rho_1), \\
\psi_\rho(r) & \text{if } r \in (\rho_1, 1). 
\end{cases} \]  

(b2)

Furthermore, an upper bound of \( \xi_\varepsilon(r, t) \) is easily derived. In fact, let \( \theta_\alpha(r) \) be the unique solution of

\[ \begin{align*}
\theta'' + \frac{n-1}{r} \theta' &= q \theta^{2q-1}, \quad r > 0, \\
\theta(0) &= \alpha, \quad \theta'(0) = 0.
\end{align*} \]

Since \( (r^{n-1} \theta_\alpha')' = qr^{n-1} \theta_\alpha^{2q-1} \), we see that \( \theta_\alpha'(r) > 0 \) for \( r > 0 \) if \( \alpha > 0 \), whence follows \( \theta_\alpha(r) > 0 \) for \( r > 0 \). Furthermore, since \( \lim_{\alpha \to 0} \theta_\alpha(r) = 0 \) uniformly for \( r \in [0, 1] \), there exists \( \alpha_0 > 0 \) such that

\[ \sup_{r \in (0,1)} \theta_{\alpha_0}(r) < \min \left\{ \inf_{r \in (0,1)} \bar{u}_0(r), 1 \right\}. \]

Since \( u_\varepsilon(1, t) \equiv \varepsilon^{-1/(q-1)} \) for \( t \in (0, \infty) \), by the comparison argument, we obtain for \( \varepsilon \in (0, 1) \)

\[ u_\varepsilon(r, t) \geq \theta_{\alpha_0}(r) \geq \alpha_0 \quad \text{in } (0, 1) \times (0, \infty), \]

which yields that for \( \varepsilon \in (0, 1) \)

\[ \xi_\varepsilon(r, t) \leq \alpha_0^{-1/(q-1)} \quad \text{in } (0, 1) \times (0, \infty). \]  

(b3)

**Lemma 5.3.** There exists \( d > 0 \) such that if \( \varepsilon \in (0, \bar{q}/2d) \), then it holds that

\[ \xi_\varepsilon(r, t) \geq \psi_\rho(r) + \varepsilon - d \varepsilon (1-r) \quad \text{in } (1-1/d, 1) \times [0, \infty). \]  

(5.4)

**Proof.** We set \( w(r) = \psi_\rho(r) + \varepsilon - d \varepsilon (1-r) \) and \( r_1(d) = 1 - 1/d \). A direct computation shows that

\[ Lw := -w_{rr} - \frac{n-1}{r} w_r - \frac{\hat{q}}{w} \left( \hat{q}^2 - w^2_r \right) \]

\[ = \hat{q} \left( \frac{1}{\psi_\rho} - \frac{1}{w} \right) \left( \hat{q}^2 - |\partial_r \psi_\rho|^2 \right) + \frac{\hat{q} \varepsilon}{w} \left( 2 \partial_r \psi_\rho + d \varepsilon \right) - \frac{(n-1)d \varepsilon}{r} \]

\[ = \frac{\hat{q} \varepsilon}{w} \left( (1-d(1-r)) \left( \frac{\hat{q}^2 - |\partial_r \psi_\rho|^2}{\psi_\rho} \right) + d \left( 2 \partial_r \psi_\rho + d \varepsilon \right) \right) - \frac{(n-1)d \varepsilon}{r}. \]

By (2.6), we can choose \( d > 0 \) large enough such that \( r_1(d) > \rho_1 \) and

\[ \bar{q} d \geq \| (\hat{q}^2 - |\partial_r \psi_\rho|^2) / \psi_\rho \|_{L^\infty(r_1,1)}, \quad -\bar{q} \leq \partial_r \psi_\rho(r) \leq -3\bar{q}/4 \quad \text{for } r \in (r_1, 1). \]
Since \( w(r) > 0 \) for \( r \in (r_1, 1) \), we can verify that

\[
Lw \leq \frac{\hat{q} \varepsilon}{w} \left( \| (\hat{q}^2 - |\partial_r \psi_\rho|^2) / \psi_\rho \|_{L^\infty(r_1, 1)} + d \left( -\frac{3\hat{q}}{2} + d \varepsilon \right) \right) \\
\leq \frac{\hat{q} \varepsilon}{w} \left( \hat{q}d + d \left( -\frac{3\hat{q}}{2} + d \varepsilon \right) \right) = -\frac{\hat{q}d \varepsilon}{w} \left( -\frac{\hat{q}}{2} + d \varepsilon \right), \quad r \in (r_1, 1).
\]

Hence we obtain

\[
Lw \leq 0, \quad r \in (r_1, 1)
\]

if \( \varepsilon \in (0, \hat{q}/2d) \). From (b2) and \( r_1 > \rho_1 \), it holds that

\[
w(r_1) = \psi_\rho(r_1) \leq \xi_\varepsilon(r_1, t), \quad t \in (0, \infty).
\]

Moreover, since \( w(1) = \varepsilon \), it is clear that

\[
w(1) = \xi_\varepsilon(1, t), \quad t \in (0, \infty).
\]

Recalling the definition of \( w(r) \), \( r_1 > \rho_1 \) and (b1), we get

\[
w(r) \leq \psi_\rho(r) + \varepsilon \leq \xi_\varepsilon(r, 0), \quad r \in (r_1, 1).
\]

Hence, applying the comparison argument in \((r_1, 1) \times (0, \infty)\), we derive \( w(r) \leq \xi_\varepsilon(r, t) \) in \((r_1, 1) \times (0, \infty)\), which completes the proof. \( \square \)

From now, we fix \( d = d_0 > 0 \) given in Lemma 5.3 and set \( \varepsilon_0 = \hat{q}/2d_0 \).

**Lemma 5.4.** There exist \( r_0 > 0 \) and \( \varepsilon_1 \in (0, \varepsilon_0) \) such that \( \partial_r \xi_\varepsilon(r, t) \leq 0 \) for \((r, t) \in (r_0, 1) \times (0, \infty)\) if \( \varepsilon \in (0, \varepsilon_1) \).

**Proof.** Since \( \xi_\varepsilon(1, t) = \varepsilon \), we obtain from Lemma 5.3

\[
\partial_r \xi_\varepsilon(1, t) \leq -(\hat{q} - d_0 \varepsilon), \quad t \in (0, \infty),
\]

which implies that

\[
\partial_r u_\varepsilon(1, t) > 0, \quad t \in (0, \infty).
\]

Let \( \#N_{t}^\varepsilon \) be the number of zeros of \( \partial_r u_\varepsilon(\cdot, t) \) in \((0, 1)\) and set \( t_0^\varepsilon = \inf \{ t \in (0, \infty); \#N_{t}^\varepsilon = 0 \} \). By Lemma 2.2 with (5.6), we find that \( \#N_{t}^\varepsilon = 0 \) for \( t > t_0^\varepsilon \). Hence it is sufficient to consider the case where \( t < t_0^\varepsilon \). By the same arguments as in the proof of Lemma 4.1, we obtain

\[
\max_{r \in [0, z^\varepsilon(t)]} |\partial_r u_\varepsilon(r, t)| \leq \max_{r \in [0, z_0^\varepsilon]} |\partial_r u_\varepsilon, 0(r)|,
\]

where $z^\varepsilon(t) = \sup\{r \in [0, 1); \partial_r u_\varepsilon(r, t) = 0\}$ and $z_0^\varepsilon = \lim_{t \to 0^+} z^\varepsilon(t)$. Since $u_{\varepsilon,0}(r) = (\bar{u}_0(r)^{-q-1} + \varepsilon)^{-1/(q-1)}$, it holds that $\partial_r u_{\varepsilon,0}(r) = (\bar{u}_0(r)^{-q-1} + \varepsilon)^{-q/(q-1)}\bar{u}_0(r)^{-q} \partial_r \bar{u}_0(r)$. Then, by Lemma 2.3, we see that $z_0^\varepsilon \leq z_T$, where $z_T \in (0, 1)$ is defined in Lemma 4.8. Hence, from (b1), there exists $c > 0$ such that
\[
\max_{r \in [0,z_0^\varepsilon]} |\partial_r u_{\varepsilon,0}(r)| \leq \max_{r \in [0,z_T]} |\partial_r u_{\varepsilon,0}(r)| \leq c, \quad t \in (0,t_0^\varepsilon).
\]
Therefore we obtain
\[
\max_{r \in [0,z(t)]} |\partial_r u_\varepsilon(r,t)| \leq c, \quad t \in (0,t_0^\varepsilon).
\]
From Proposition 4.10, we recall that if $\alpha \geq \alpha_1$
\[
u(r,t) \geq \chi_\alpha(r), \quad (\zeta_\alpha, 1) \times (t_\alpha, T).
\]
Since $u(r,t)$ is uniformly bounded on $(0,R) \times (0,T)$ for any $R \in (\zeta_\alpha, 1)$, by the parabolic regularity theory, $u(r,t)$ can be continuously extended to $t = T$ as a solution, more precisely $u \in C^{2,1}([0,R] \times (0,T])$ for any $R \in (\zeta_\alpha, 1)$. Therefore, since $\bar{u}_0(r) = u(r,T)$, by the strong maximum principle it holds that
\[
\bar{u}_0(r) > \chi_\alpha(r), \quad r \in (\zeta_\alpha, 1)
\]
provided that $\alpha \geq \alpha_1$. Here we claim that for any $\alpha \geq \alpha_1$ there exists $\varepsilon_\alpha > 0$ such that
\[
u_{\varepsilon,0}(r) \geq \chi_\alpha(r), \quad r \in (\zeta_\alpha, 1) \quad \text{for all } \varepsilon \in (0,\varepsilon_\alpha).
\]
In fact, let $l \in (\zeta_\alpha, 1)$ be the point such that $\chi_{4\alpha}(l) = 2\alpha$. Then, since $u_{\varepsilon,0}(r) = (\bar{u}_0(r)^{-q-1} + \varepsilon)^{-1/(q-1)}$ and $\chi_{4\alpha}'(r) > 0$, by (5.7), we get
\[
u_{\varepsilon,0}(r) \geq ((2\alpha)^{-q-1} + \varepsilon)^{-1/(q-1)}, \quad r \in (l,1).
\]
Therefore there exists $\varepsilon_\alpha > 0$ such that $u_{\varepsilon,0}(r) \geq 3\alpha/2$ for all $r \in (l,1)$ and $\varepsilon \in (0,\varepsilon_\alpha)$. Since $0 < \chi_\alpha(r) < \alpha$ in $(\zeta_\alpha, 1)$, we find that $u_{\varepsilon,0}(r) \geq \chi_\alpha(r)$ for all $r \in (l,1)$ and $\varepsilon \in (0,\varepsilon_\alpha)$. Next we verify that (5.8) holds for all $r \in (\zeta_\alpha, 1)$. It is clear that it suffices to consider the case $l \in (\zeta_\alpha, 1)$. Since $\bar{u}_0(r) \in C([0,1])$, by (5.7), it is easy to see that $u_{\varepsilon,0}(r) \geq \chi_\alpha(r)$ for $(\chi_\alpha(r), l)$ for sufficiently small $\varepsilon > 0$. Thus the claim is verified. The rest of proof can be done by the same arguments as in the proof of Lemma 4.2, which completes the proof. \hfill \square

Here we provide time global $L^\infty$-estimates for $\partial_r u_\varepsilon, \partial_t u_\varepsilon$ and $\partial_{rr} u_\varepsilon$. 

Lemma 5.5. For any $p \in (1, \infty)$ and $R \in (0, 1)$, there exists $\bar{K} = \bar{K}(p, R) > 0$ such that

$$\sup_{(r,t) \in (0,R) \times (0,\infty)} \left( |\partial_t u_\varepsilon(r,t)| + |\partial_r u_\varepsilon(r,t)| \right) + \sup_{t \in (0,\infty)} \|\partial_{rr} u_\varepsilon(t)\|_{L_1^p(0,R)} \leq \bar{K}.$$ 

Proof. In this proof, we use both Cartesian coordinates $(x, t)$ and the polar coordinates $(r, t)$. Here we write $U_\varepsilon(x,t) = u_\varepsilon(|x|, t)$ and $U_\varepsilon,0(x) = u_\varepsilon,0(|x|)$. From (b2), for any $R \in (0, 1)$ there exists $C_R > 0$ such that

$$\|U_\varepsilon\|_{L_\infty(B_R \times (0,\infty))} \leq C_R. \tag{5.9}$$

Hence, by the parabolic regularity theory, for $R \in (0, 1)$ there exists $K_R > 0$ such that

$$\sup_{(x,t) \in B_R \times (1,\infty)} \left( U_\varepsilon(x,t) + |\nabla U_\varepsilon(x,t)| + \sum_{i,j=1}^n |\partial_i \partial_j U_\varepsilon(x,t)| + |\partial_i U_\varepsilon(x,t)| \right) \leq K_R,$$

where $\partial_i = \partial / \partial x_i \ (i = 1, \ldots, n)$. To complete the proof, it suffices to establish estimates for $t \in (0, 1)$. First we derive the $L^1$-estimates of $\partial_r u_\varepsilon$. We set $w_\varepsilon(r,t) = \partial_r u_\varepsilon(r,t)$ and $w_\varepsilon,0(r) = \partial_r u_\varepsilon,0(r)$. Then $w_\varepsilon(r,t)$ satisfies

$$\begin{cases}
\partial_t w_\varepsilon = \partial_{rr} w_\varepsilon + \frac{n-1}{r} \partial_r w_\varepsilon - \frac{n-1}{r^2} w_\varepsilon - q(2q-1)u_\varepsilon^{2q-2} w_\varepsilon \quad &\text{in } (0, 1) \times (0, \infty), \\
w_\varepsilon(r,0) = w_\varepsilon,0(r) \quad &\text{in } (0, 1).
\end{cases}$$

From Lemma 4.8 and (b1), there exists $H_0 > 0$ such that

$$\inf_{r \in (0,1)} w_\varepsilon,0(r) \geq -H_0.$$

Moreover, from Lemma 5.4, we note that $w_\varepsilon(1,t) > 0$ for $t \in (0, \infty)$. Hence $\tilde{w}(r,t) \equiv -H_0$ is a sub-solution of the equation above. As a consequence, by the comparison argument in $B_1 \times (0, \infty)$, we obtain

$$\inf_{(r,t) \in (0,1) \times (0,\infty)} w_\varepsilon(r,t) \geq -H_0.$$ 

Therefore, since $[\partial_r u_\varepsilon]_+ = \partial_r u_\varepsilon + [\partial_r u_\varepsilon]_-$ and $[w_\varepsilon]_- \leq H_0$, we see that

$$\int_0^R [w_\varepsilon]_+(r,t)dr = \int_0^R \left( \partial_r u_\varepsilon(r,t) + [w_\varepsilon]_-(r,t) \right) dr \leq u_\varepsilon(R,t) + H_0.$$ 

Thus, from (5.9), there exists $H_1 = H_1(R) > 0$ such that for $t \in (0, \infty)$

$$\|\partial_r u_\varepsilon(t)\|_{L_1^1(0,R)} = \|w_\varepsilon(t)\|_{L_1^1(0,R)} \leq H_1. \tag{5.10}$$
We set \( Z^i_\varepsilon(x,t) = \partial_t U_\varepsilon(x,t) \) and \( Z^i_\varepsilon,0(x) = \partial_t U_\varepsilon,0(x) \). Then \( Z^i_\varepsilon(x,t) \) is a solution of

\[
\begin{cases}
\partial_t Z^i_\varepsilon = \Delta Z^i_\varepsilon - q(2q - 1)U^2q-2_\varepsilon Z^i_\varepsilon & \text{in } B_1 \times (0, \infty), \\
Z^i_\varepsilon(x,0) = Z^i_\varepsilon,0(x) & \text{in } B_1.
\end{cases}
\]

Theorem 6.30 in [18] together with (5.9) assures that for \( 0 < R_1 < R < 1 \), there exists \( H_2 = H_2(R, R_1) > 0 \) such that

\[
\sup_{(x,t) \in B_{R_1} \times (0,1)} |Z^i_\varepsilon(x,t)| \leq H_2 \left( \int_0^1 dt \int_{B_R} |Z^i_\varepsilon(x,t)| \, dx + \|Z^i_\varepsilon,0\|_{L^\infty(B_R)} \right).
\]

Since \( \|Z^i_\varepsilon\|_{L^1(B_R)} \sim \|\partial_t u_\varepsilon\|_{L^1(0,R)} \), from (5.10) and (b1), there exists \( H_3 = H_3(R, R_1) \) such that

\[
\sup_{(x,t) \in B_{R_1} \times (0,1)} |\nabla U_\varepsilon(x,t)| \leq H_3. \tag{5.11}
\]

Next we derive the boundedness of \( \partial_t u_\varepsilon \). Let \( 0 < R_2 < R_1 < 1 \) and \( \theta(x) \) be a smooth cut off function such that \( \theta(x) = 1 \) if \( x \in B_{R_2} \) and \( \theta(x) = 0 \) if \( |x| \geq R_1 \). Multiplying \( \partial_t U_\varepsilon = \Delta U_\varepsilon - qU^2q-1_\varepsilon \) by \( (\partial_t U_\varepsilon)\theta^2 \) and integrating over \( B_1 \times (0,1) \), we get

\[
\int_0^1 dt \int_{B_1} |\partial_t U_\varepsilon|^2 \theta^2 \, dx = -\frac{1}{2} \left[ \int_{B_1} |\nabla U_\varepsilon|^2 \theta^2 \, dx + \int_{B_1} U^2q_\varepsilon \theta^2 \, dx \right]_{t=0}^{t=1} - \int_0^1 dt \int_{B_1} \nabla U_\varepsilon \cdot (\nabla \theta^2) \partial_t U_\varepsilon \, dx.
\]

Hence, by the Schwarz inequality, there exists \( H_4 = H_4(R_1, R_2) > 0 \) such that

\[
\int_0^1 dt \int_{B_{R_2}} |\partial_t U_\varepsilon|^2 \, dx \leq H_4 \left( \|\nabla U_{\varepsilon,0}\|^2_{L^2(B_{R_1})} + \|U_\varepsilon\|^2_{L^\infty(B_{R_1} \times (0,1))} + \|\nabla U_\varepsilon\|^2_{L^\infty(B_{R_1} \times (0,1))} \right).
\]

Therefore, from (5.9), (5.11) and (b1), there exists \( H_5 = H_5(R, R_1, R_2) > 0 \) such that

\[
\int_0^1 dt \int_{B_{R_2}} |\partial_t U_\varepsilon|^2 \, dx \leq H_5. \tag{5.12}
\]

Set \( Y_\varepsilon(x,t) := \partial_t U_\varepsilon(x,t) \). Then \( Y_\varepsilon \in C([0, \infty); L^p(B_1)) \) (\( 2 \leq p < \infty \)) is a solution of

\[
\begin{cases}
\partial_t Y_\varepsilon = \Delta Y_\varepsilon - q(2q - 1)U^{2q-2}_\varepsilon Y_\varepsilon & \text{in } B_1 \times (0, \infty), \\
Y_\varepsilon(x,0) = Y_{\varepsilon,0}(x) := \Delta U_{\varepsilon,0}(x) - qU_{\varepsilon,0}(x)^{2q-1} & \text{in } B_1.
\end{cases}
\]
Therefore Theorem 6.30 in [18] together with (5.9) assures that for $0 < R_3 < R_2 < 1$, there exists $H_6 = H_6(R_2, R_3) > 0$ such that

$$\sup_{(x,t) \in B_{R_3} \times (0,1)} |Y_\varepsilon(x,t)|^2 \leq H_6 \left( \int_0^1 dt \int_{B_{R_2}} |Y_\varepsilon(x,t)|^2 dx + \|Y_\varepsilon,0\|_{L^\infty(B_{R_2})}^2 \right).$$

Hence, from (5.12) and (b1), there exists $H$ such that

$$H \sup_{(x,t) \in B_{R_3} \times (0,1)} |\partial_t U_\varepsilon(x,t)|^2 \leq H_7. \quad (5.13)$$

Finally we derive the boundedness of $\partial_i \partial_j U_\varepsilon(x,t)$ ($1 \leq i, j \leq n$). By the elliptic regularity theory, for $p \in (1, \infty)$ and $0 < R_4 < R_3 < 1$ there exists $H_8 = H_8(p, R_3, R_4) > 0$ such that for $t \in (0,1)$

$$\sum_{i,j=1}^n \|\partial_i \partial_j U_\varepsilon(t)\|_{L^p(B_{R_4})} \leq H_8 \left( \|\partial_t U_\varepsilon(t) + qU_\varepsilon(t)^{2q-1}\|_{L^p(B_{R_3})} + \|U_\varepsilon(t)\|_{L^p(B_{R_3})} \right). \quad (5.14)$$

Since

$$\partial_t u_\varepsilon(|x|, t) = \partial_t U_\varepsilon(x,t), \quad |\partial_r u_\varepsilon(|x|, t)| = |\nabla U_\varepsilon(x,t)|,$$

$$|\partial_{rr} u_\varepsilon(|x|, t)| \leq \sum_{i,j=1}^n |\partial_j \partial_i U_\varepsilon(x,t)|,$$

from (5.9), (5.11), (5.13) and (5.14), we can derive the conclusion. \hfill \square

**Lemma 5.6.** There exist $b > 0$ and $\varepsilon_2 \in (0, \varepsilon_1)$ such that

$$\partial_r \xi_\varepsilon(r,t) \leq -((q-d_0\varepsilon) + b(1-r)) \quad \text{in } (1 - (q-d_0\varepsilon)/b, 1) \times (0, \infty) \quad \text{for all } \varepsilon \in (0, \varepsilon_2).$$

**Proof.** We set $V_\varepsilon(r) = -((q-d_0\varepsilon) + b(1-r))$ and $r_\varepsilon(b) = 1 - ((q-d_0\varepsilon)/b)$. Since $\lim_{b \to \infty} r_\varepsilon(b) = 1$ uniformly for $\varepsilon \in (0, \varepsilon_1)$, we can assume that

$$r_\varepsilon > \max\{1 - 1/d_0, r_0, 1 - \delta \rho\},$$

where $r_0 \in (0,1)$ is given in Lemma 5.4. Since $V_\varepsilon(r) < 0$ in $(r_\varepsilon, 1)$, it is verified that

$$L_r V_\varepsilon \geq \frac{(n-1)b}{r} - \frac{(n-1)q}{r^2} + \frac{\hat{q}|V_\varepsilon|}{\xi_\varepsilon^2} \left( V_\varepsilon^2 - q^2 + 2b\xi_\varepsilon \right)$$

$$\geq \frac{(n-1)(r_\varepsilon b - q)}{r^2} + \frac{\hat{q}|V_\varepsilon|}{\xi_\varepsilon^2} \left( V_\varepsilon^2 - q^2 + 2b\xi_\varepsilon \right), \quad r \in (r_\varepsilon, 1).$$
From (5.4), we see that
\[
V_ε^2 - q^2 + 2bξ_ε = -2q d_0 η + d_0^2 η^2 - 2(q - d_0 η)b(1 - r) + b^2(1 - r)^2 + 2bξ_ε
\geq 2ε(b - q d_0) + 2b\{ψ_ρ - q(1 - r)\} + b^2(1 - r)^2, \quad r \in (r_ε, 1).
\]
By (2.6) and \(r_ε > 1 - δ_ρ\), we note that \(ψ_ρ(r) \geq q(1 - r) - κ(1 - r)^2\) in \((r_ε, 1)\). Hence we get
\[
V_ε^2 - q^2 + 2bξ_ε \geq 2ε(b - q d_0) + b \{ψ_ρ - q(1 - r)\} + b^2(1 - r)^2, \quad r \in (r_ε, 1).
\]
Therefore we obtain
\[
L_r V_ε \geq 0 \text{ in } (r_ε, 1) \text{ if } b > \frac{q}{r_0} + \bar{q} d_0 + 2κ. By (5.5), it is clear that \(V_ε(1) \geq \partial_r ξ_ε(1, t)\) for \(t \in (0, ∞)\). From Lemma 5.4 and \(V_ε(r_ε) = 0\), we find that \(V_ε(r_ε) \geq \partial_r ξ_ε(r_ε, t)\) for \(t \in (0, ∞)\). Moreover, from (b1), we note that
\[
\partial_r \bar{v}_0(r) \leq -\bar{q} + \bar{c}(1 - r).
\]
Hence we obtain \(V_ε(r) \geq \partial_r \bar{v}_0(r) = \partial_r ξ_ε(r, 0)\) if \(b \geq \bar{c}\). Thus, by the comparison argument, we conclude that \(V_ε(r) \geq \partial_r ξ_ε(r, t)\) in \((r_ε, 1) \times (0, ∞)\), which completes the proof. □

Here we fix a constant \(b = b_0 > 0\) given in Lemma 5.6 such that \(b_0 > 2κ\). Moreover we set
\[
\bar{r}_ε = 1 - (q - 2d_0 η)/2b_0, \quad Q_ε = (\bar{r}_ε, 1) \times (0, ∞).
\]
Then, from Lemma 5.6, we see that
\[
\partial_r ξ_ε(r, t) \leq -\bar{q}/2 \text{ in } Q_ε. \quad (5.15)
\]
Taking \(b_0 > 0\) large enough if necessary, we can assume that
\[
\bar{r}_ε > \max\{1 - 1/d_0, r_0, 1 - δ_ρ\},
\]
where \(r_0 \in (0, 1)\) is given in Lemma 5.4.

**Lemma 5.7.** There exist \(\bar{c}_1\) and \(γ > 0\) such that
\[
\partial_t u_ε(r, t) \leq \bar{c}_1 e^{γt} \partial_r u_ε(r, t) \text{ in } Q_ε \text{ for all } ε \in (0, ε_2).
\]

**Proof.** We set
\[
c_∞ = \sup_{r ∈ (0, 1)} \left( |\partial_r \bar{v}_0(r)| + \frac{n - 1}{r} |\partial_r \bar{v}_0(r)| + \frac{\bar{q}}{\bar{v}_0(r)} |q^2 - |\partial_r \bar{v}_0(r)|^2| \right).
\]
Here since \( \bar{v}_0(1) = 0 \), \( \partial_r \bar{v}_0(1) = \bar{q} \) and \( v_0 \in W^2_r(0, 1) \), we obtain
\[
|v_0(r)| = \bar{q}(1 - r) + O((1 - r)^2), \quad |\partial_r v_0(r)| = \bar{q} + O(1 - r).
\]
Therefore we can deduce that \( c_\infty < \infty \). By (5.15), it holds that
\[
m_1|\partial_r \xi_\varepsilon(r, 0)| \geq c_\infty, \quad r \in (\bar{r}_\varepsilon, 1),
\]
where \( m_1 = 2c_\infty/\bar{q} \). From \( \partial_t \xi_\varepsilon \in C([0, \infty); L^p_r(0, 1)) \) for any \( p \in [1, \infty) \), we find that
\[
\partial_t \xi_\varepsilon(r, 0) = \partial_{rr} \bar{v}_0(r) + \frac{n - 1}{r} \partial_r \bar{v}_0(r) + \frac{\hat{q}}{(\bar{v}_0(r) + \varepsilon)} (\bar{q}^2 - |\partial_r \bar{v}_0(r)|^2), \quad r \in (0, 1).
\]
Hence, by the definition of \( c_\infty \), we see that \( \|\partial_t \xi_\varepsilon(0)\|_\infty \leq c_\infty \). Therefore, by (5.16), we obtain \( |\partial_t \xi_\varepsilon(r, 0)| \leq m_1 |\partial_r \xi_\varepsilon(r, 0)| \) for \( r \in (\bar{r}_\varepsilon, 1) \), which implies
\[
\partial_t u_\varepsilon(r, 0) \leq m_1 \partial_r u_\varepsilon(r, 0), \quad r \in (\bar{r}_\varepsilon, 1).
\]
By (5.15), we note that \( \partial_t u_\varepsilon(\bar{r}_\varepsilon, t) \geq u_\varepsilon(\bar{r}_\varepsilon, t)^q/2 \) for \( t \in (0, \infty) \). Hence, since \( \inf_{t \in (0, \infty)} u_\varepsilon(\bar{r}_\varepsilon, t) \geq \delta \) for some \( \delta > 0 \) (see (b3)), by (5.6) we get \( \inf_{t \in (0, \infty)} \partial_t u_\varepsilon(\bar{r}_\varepsilon, t) \geq \delta_1 > 0 \). Furthermore, from Lemma 5.5, it holds that \( \sup_{t \in (0, \infty)} |\partial_t u_\varepsilon(\bar{r}_\varepsilon, t)| \leq \delta_2 \) for some \( \delta_2 > 0 \). Hence there exists \( m_2 > 0 \) independent of \( \varepsilon > 0 \) such that
\[
\partial_t u_\varepsilon(\bar{r}_\varepsilon, t) \leq m_2 \partial_r u_\varepsilon(\bar{r}_\varepsilon, t), \quad t \in (0, \infty).
\]
Since \( \partial_t u_\varepsilon(1, t) = 0 \) for \( t \in (0, \infty) \), it is clear that
\[
\partial_t u_\varepsilon(1, t) \leq \partial_r u_\varepsilon(1, t), \quad t \in (0, \infty).
\]
Thus we can apply the same comparison arguments as in the proof of Lemma 4.4 and obtain the conclusion. \( \square \)

**Lemma 5.8.** There exists \( \bar{c}_2 > 0 \) such that if \( \varepsilon \in (0, \varepsilon_2) \)
\[
\partial_r \xi_\varepsilon(r, t) \geq -\bar{c}_2 \varepsilon^{\gamma t} \quad \text{in } Q_\varepsilon.
\]

**Proof.** By Lemma 5.3 and Lemma 5.6, we see that
\[
\frac{|\partial_r \xi_\varepsilon|^2 - \bar{q}^2}{\xi_\varepsilon} \geq \frac{-2\bar{q}(d_0 \varepsilon + b_0(1 - r))}{\xi_\varepsilon} \geq \frac{-2\bar{q}(d_0 \varepsilon + b_0(1 - r))}{\psi_\rho + \varepsilon - d_0 \varepsilon(1 - r)}, \quad r \in (\bar{r}_\varepsilon, 1).
\]
Hence, since \( \psi_\rho(r) \geq \bar{q}(1 - r) - \kappa(1 - r)^2 \), it holds by (b) that for \( r \in (\bar{r}_\varepsilon, 1) \),
\[
\frac{|\partial_r \xi_\varepsilon|^2 - \bar{q}^2}{\xi_\varepsilon} \geq \frac{-2\bar{q}(d_0 \varepsilon + b_0(1 - r))}{\varepsilon + (\bar{q} - \kappa(1 - r) - d_0 \varepsilon)(1 - r)} + \frac{-2\bar{q}b_0(1 - r)}{\varepsilon + (\bar{q} - \kappa(1 - r) - d_0 \varepsilon)(1 - r)}.
\]
Thus we conclude that
\[ q - \kappa(1 - r) - d_0 \varepsilon > \bar{q}/2 - \kappa(1 - r) \]
for \( \varepsilon \in (0, \varepsilon_2) \). For the rest of the proof, we assume \( \varepsilon \in (0, \varepsilon_2) \). Hence, using the definition of \( \bar{r} \) and the relation \( b_0 > 2\kappa \), we obtain
\[ \bar{q} - \kappa(1 - r) - d_0 \varepsilon > \bar{q}/2 - \kappa(1 - r) > \bar{q}/4, \quad r \in (\bar{r}, 1). \]
Hence we obtain
\[
\frac{|\partial_r \xi_\varepsilon|^2 - \bar{q}^2}{\xi_\varepsilon} \geq -2(\bar{q}d_0 + 4b_0) \quad \text{in } Q_\varepsilon. \tag{5.17}
\]
For simplicity, we set \( D_0 = 2 \bar{q} (\bar{q}d_0 + 4b_0) \). Since \( \xi_\varepsilon(r, t) \) is a solution of (5.3), we see that
\[
\partial_{rr} \xi_\varepsilon = \partial_t \xi_\varepsilon - \frac{n-1}{r} \partial_r \xi_\varepsilon + \frac{\bar{q}}{\xi_\varepsilon} (|\partial_r \xi_\varepsilon|^2 - \bar{q}^2) \geq \partial_t \xi_\varepsilon - D_0 \quad \text{in } Q_\varepsilon,
\]
which yields
\[
[\partial_{rr} \xi_\varepsilon]_- \leq [\partial_t \xi_\varepsilon]_- + D_0 \quad \text{in } Q_\varepsilon.
\]
Hence, using the fact that \( \partial_t \xi_\varepsilon \leq 0 \) in \( Q_\varepsilon \) and Lemma 5.7, we get
\[
\int_{\bar{r}}^1 [\partial_{rr} \xi_\varepsilon]_- \leq \int_{\bar{r}}^1 ([\partial_t \xi_\varepsilon]_- + D_0) \leq \int_{\bar{r}}^1 (\bar{c}_1 e^{\gamma t} |\partial_r \xi_\varepsilon| + D_0)
= \int_{\bar{r}}^1 (\bar{c}_1 e^{\gamma t} \partial_r \xi_\varepsilon + D_0) \leq \bar{c}_1 e^{\gamma t} \xi_\varepsilon (\bar{r}, t) + D_0, \quad t \in (0, \infty). \tag{5.18}
\]
On the other hand, using again the fact that \( \partial_r \xi_\varepsilon \leq 0 \) in \( Q_\varepsilon \), we have
\[
\int_{\bar{r}}^1 [\partial_{rr} \xi_\varepsilon]_+ = \int_{\bar{r}}^1 (\partial_{rr} \xi_\varepsilon + [\partial_{rr} \xi_\varepsilon]_-)
\leq [\partial_r \xi_\varepsilon (\bar{r}, t)] + \bar{c}_1 e^{\gamma t} \xi_\varepsilon (\bar{r}, t) + D_0, \quad t \in (0, \infty). \tag{5.19}
\]
Hence (5.18) and (5.19) give
\[
\int_{\bar{r}}^1 |\partial_{rr} \xi_\varepsilon| \leq \int_{\bar{r}}^1 ([\partial_{rr} \xi_\varepsilon]_+ + [\partial_{rr} \xi_\varepsilon]_-)
\leq [\partial_r \xi_\varepsilon (\bar{r}, t)] + 2\bar{c}_1 e^{\gamma t} \xi_\varepsilon (\bar{r}, t) + 2D_0, \quad t \in (0, \infty).
\]
Thus we conclude that
\[
|\partial_r \xi_\varepsilon (r, t)| \leq [\partial_r \xi_\varepsilon (\bar{r}, t)] + \int_{\bar{r}}^r |\partial_{rr} \xi_\varepsilon|
\leq 2[\partial_r \xi_\varepsilon (\bar{r}, t)] + 2\bar{c}_1 e^{\gamma t} \xi_\varepsilon (\bar{r}, t) + 2D_0 \quad \text{in } Q_\varepsilon.
\]
From (b3) and Lemma 5.5, we note that \( \xi_\varepsilon (\bar{r}, t) \) and \( \partial_r \xi_\varepsilon (\bar{r}, t) \) are uniformly bounded for \( t \in (0, \infty) \). Thus the proof is completed. \( \square \)
Lemma 5.9. There exists $\tilde{c}_3 > 0$ such that

$$|\partial_r \xi_\varepsilon(r,t)|^2 - \hat{q}^2 \leq \tilde{c}_3 e^{2\gamma t} \xi_\varepsilon(r,t) \quad \text{in } Q_\varepsilon \text{ for all } \varepsilon \in (0,\varepsilon_2).$$

Proof. Repeating the same arguments as in the proof of Lemma 4.6 with the aid of Lemma 5.7 and Lemma 5.8, we obtain

$$\partial_r u_\varepsilon(r,t)^2 - u_\varepsilon(r,t)^{2q} \leq \partial_r (\tilde{r}_\varepsilon, t)^2 + ce^{2\gamma t} u_\varepsilon(r,t)^{q+1} \quad \text{in } Q_\varepsilon$$

for some $c > 0$. From Lemma 5.5, there exists $c' > 0$ such that

$$\partial_r u_\varepsilon(\tilde{r}_\varepsilon, t) \leq c', \quad t \in (0,\infty).$$

Hence, by using the relation $|\partial_r \xi_\varepsilon|^2 = \hat{q}^2 (\partial_r u_\varepsilon)^2 / u_\varepsilon^{2q}$, we can show that there exists $c'' > 0$ such that

$$|\partial_r \xi_\varepsilon(r,t)|^2 - \hat{q}^2 \leq c'' \left( u_\varepsilon(r,t)^{-(q+1)} + e^{2\gamma t}\right) \xi_\varepsilon(r,t) \quad \text{in } Q_\varepsilon,$$

which completes the proof. \qed

Lemma 5.10. For any $p \in (1,\infty)$ and $\tau > 0$ there exists $\tilde{c}_{p,\tau} > 0$ such that

$$\int_0^\tau \int_0^1 (|\partial_t \xi_\varepsilon(r,t)|^p + |\partial_{rr} \xi_\varepsilon(r,t)|^p) r^{n-1} dr \leq \tilde{c}_{p,\tau} \quad \text{for all } \varepsilon \in (0,\varepsilon_2).$$

Proof. We set $\eta_\varepsilon(x,t) = \xi_\varepsilon(|x|,t) - \xi_\varepsilon,0(|x|)$. Then $\eta_\varepsilon(x,t)$ satisfies

$$\begin{cases}
\partial_t \eta_\varepsilon = \Delta \eta_\varepsilon + F_\varepsilon(x,t) & \text{in } B_1 \times (0,\infty), \\
\eta_\varepsilon = 0 & \text{on } \partial B_1 \times (0,\infty) \cup B_1 \times \{0\},
\end{cases}$$

where $F(x,t)$ is given by

$$F_\varepsilon(x,t) = \frac{\hat{q}}{\xi_\varepsilon(|x|,t)} (\hat{q}^2 - |\partial_r \xi_\varepsilon(|x|,t)|^2) + \Delta \xi_\varepsilon,0(|x|).$$

Then, by the $L^p$-regularity theory, for $p \in (1,\infty)$ and $\tau > 0$ there exists $c_0 = c_0(p,\tau) > 0$ such that

$$\int_0^\tau \int_{B_1} \left( |\partial_t \eta_\varepsilon(x,t)|^p + \sum_{i,j=1}^n |\partial_i \partial_j \eta_\varepsilon(x,t)|^p \right) dx dt \leq c_0 \int_0^\tau \int_{B_1} (|F_\varepsilon(x,t)|^p + |\xi_\varepsilon(x,t)|^p) dx dt. \quad (5.20)$$
From (5.17) and Lemma 5.9, there exists \( D_1 > 0 \) such that
\[
\frac{||\partial_r \xi_\varepsilon(r, t)||^2 - \bar{q}^2}{\xi_\varepsilon(r, t)} \leq D_1 e^{2\gamma t} \quad \text{in } Q^\varepsilon. \tag{5.21}
\]

Moreover, from (b2) and Lemma 5.5, there exists \( D_2 > 0 \) such that
\[
\frac{||\partial_r \xi_\varepsilon(r, t)||^2 - \bar{q}^2}{\xi_\varepsilon(r, t)} \leq D_2 \quad \text{in } (0, 1) \times (0, \infty) \setminus Q^\varepsilon. \tag{5.22}
\]

Since \( \xi_{\varepsilon, 0}(r) = \bar{v}_0(r) + \varepsilon \), we observe that
\[
\Delta \xi_{\varepsilon, 0}(|x|) = \Delta \bar{v}_0(|x|).
\]

Therefore there exists \( c > 0 \) such that
\[
|F_\varepsilon(x, t)| \leq c \left( D_1 e^{2\gamma t} + D_2 + \|\bar{v}_0\|_{W^{2,\infty}(0, 1)} \right).
\]

Thus (5.20) and (b3) assure the desired estimate.

**Proof of Proposition 5.1.** First we provide interior estimates of \( \xi_\varepsilon(r, t) \). From (b2), for every \( R \in (0, 1) \) there exists \( \nu_R > 0 \) such that \( \xi_\varepsilon(r, t) \geq \nu_R \) for \( (r, t) \in (0, R) \times (0, \infty) \). This implies that \( \xi_\varepsilon(r, t) \leq \nu_R^{-1/(q-1)} \) for \( (r, t) \in (0, R) \times (0, \infty) \).

Therefore, by the standard parabolic regularity theory, we find that the sequence \( \{\xi_\varepsilon(r, t)\}_{\varepsilon > 0} \) is compact in \( C^{2,1}([\delta, \delta^{-1}]^2) \) for any \( \delta > 0 \). Since \( \xi_\varepsilon(r, t) = u_\varepsilon(r, t)^q \), by virtue of (b2), we find that the sequence \( \{\xi_\varepsilon(r, t)\}_{\varepsilon > 0} \) is compact in \( C^{2,1}([\delta, \delta^{-1}]^2) \) for any \( \delta > 0 \).

Next we provide global estimates of \( \xi_\varepsilon(r, t) \). Since \( \partial_t \xi_\varepsilon(r, t) \leq 0 \) in \( Q^\varepsilon \), from Lemma 5.8, it holds that
\[
|\partial_r \xi_\varepsilon(r, t)| \leq c_2 e^{\gamma t} \quad \text{in } Q^\varepsilon.
\]

Hence, from Lemma 5.5, there exists \( c > 0 \) such that
\[
|\partial_r \xi_\varepsilon(r, t)| \leq c e^{\gamma t} \quad \text{in } (0, 1) \times (0, \infty). \tag{5.23}
\]

Therefore, by virtue of (b3) and Lemma 5.10 and the above interior estimates, there exist a sequence \( \{\xi_{\varepsilon_k}\}_{k=1}^\infty \) and \( \xi \in C([0, 1] \times [0, \infty)) \cap C^{2,1}([0, 1] \times (0, \infty)) \) such that for any \( R \in (0, 1) \) and \( \tau > 0 \)
\[
\xi_{\varepsilon_k} \to \xi \quad \text{strongly in } C([0, 1] \times [0, \tau]),
\]
\[
\partial_t \xi_{\varepsilon_k}, \partial_{rr} \xi_{\varepsilon_k} \to \partial_t \xi, \partial_{rr} \xi \quad \text{weakly in } L^p((0, 1) \times (0, \tau)),
\]
\[
\partial_r \xi_{\varepsilon_k}(r, t) \to \partial_r \xi(r, t) \quad \text{uniformly on } [0, R] \times [\tau^{-1}, \tau].
\]
Moreover, by (5.21), (5.22), (5.23) and Lemma 5.6, we find that \( \partial_r \xi \in L^\infty((0, 1) \times (0, \tau)) \) for \( \tau > 0 \) and
\[
\xi_r(r, t) \leq 0 \quad \text{in } Q_0, \quad |\xi(r, t)|^2 - q^2 \leq ce^{2\gamma t} \xi(r, t) \quad \text{in } Q_0,
\] (5.24)
where \( Q_0 = (1 - \bar{q}/2b_0) \times (0, \infty) \). Then it is clear that
\[
\xi(r, 0) = \bar{v}_0(r) \quad \text{for } r \in (0, 1), \quad \xi(1, t) \equiv 0 \quad \text{for } t \in (0, \infty),
\]
which assures (C). Furthermore, by the mean value theorem and the fact that \( \xi(1, t) = 0 \) for \( t \in (0, \infty) \), we get
\[
\xi(r, t) \leq \|\xi(t)|\_{\infty}(1 - r) \quad \text{in } (0, 1) \times (0, \infty).
\]
Hence, from (5.24), we have
\[
|\xi_r(r, t) + \bar{q}| \leq \frac{ce^{2\gamma t}}{|\bar{q} - \xi(r, t)|} \|\xi(t)|\_{\infty}(1 - r) \leq \frac{ce^{2\gamma t}}{q} \|\xi(t)|\_{\infty}(1 - r) \quad \text{in } Q_0.
\]
Therefore, since \( \partial_r \xi \in L^\infty((0, 1) \times (0, \tau)) \) for \( \tau > 0 \), there exists \( c_r > 0 \) such that
\[
|\xi_r(r, t) + \bar{q}| \leq c_r (1 - r) \quad \text{in } Q_0 \cap \{t \in (0, \tau)\}.
\] (5.25)

Finally \( \partial_r \xi \in C([0, 1] \times (0, \infty)) \) together with (5.25), assures \( \partial_r \xi \in C([0, 1] \times (0, \infty)) \), which implies (B). Thus the proof is completed. \( \square \)

**Proof of Theorem 1.3.** Let \( v(r, t) \) be a solution of (1.2), \( T > 0 \) be its vanishing time and \( \xi(r, t) \) be the solution of (5.1) constructed in Proposition 5.1. We define \( \bar{v}(r, t) \) by (5.2). Then it is clear that \( \bar{v} \in C([0, 1] \times [0, \infty)) \). Set \( \bar{g}(r, t) = v_t(r, t) \) if \( t \in (0, T) \) and \( \bar{g}(r, t) = \xi_t(r, t) \) if \( t \in (T, \infty) \). Then, by continuity of \( \bar{v} \), it holds that
\[
\int_0^\infty dt \int_0^1 \bar{v}(r, t) \varphi_t(r, t) r^{n-1} dr = - \int_0^\infty dt \int_0^1 \bar{g}(r, t) \varphi(r, t) r^{n-1} dr
\]
for any \( \varphi \in C^\infty([0, 1] \times [0, \infty)) \) with \( \text{supp } \varphi \subset [0, 1] \times (0, \infty) \). Hence it follows that \( \partial_t \bar{v} \equiv \bar{g} \) in the distribution sense. We set \( \bar{u}(r, t) = \bar{v}(r, t)^{-1/(q-1)} \) and \( \bar{U}(x, t) = \bar{u}(|x|, t) \). The regularity of \( \bar{v} \) implies that
\[
\bar{U} \in C(B_1 \times (0, \infty)), \quad \bar{U}_i, \bar{U}_{jk}, \bar{U}_t \in L^p(B_R \times (0, \tau)) \quad (1 \leq i, j, k \leq n)
\]
for any \( p \in (1, \infty), R \in (0, 1) \) and \( \tau \in (0, \infty) \). Then \( \bar{U} \) satisfies
\[
\bar{U}_t = \Delta \bar{U}_{rr} - q \bar{U}^{2q-1} \quad \text{in } B_1 \times (0, \infty).
\]
Let $R \in (0,1)$ and $\mu_R(x,t) \in C(B_R \times [0,\infty)) \cap C^{2,1}(B_R \times (0,\infty))$ be the unique solution of
\[
\begin{cases}
\mu_t = \Delta \mu & \text{in } B_R \times (0,\infty), \\
\mu = \bar{u} & \text{on } \partial B_R \times (0,\infty), \\
\mu(x,0) = u_0(|x|) & \text{in } B_R.
\end{cases}
\]
Then $\nu_R(x,t) := \bar{U}(x,t) - \mu_R(x,t)$ satisfies
\[
\begin{cases}
\nu_t = \Delta \nu - q\bar{U}^{2q-1} & \text{in } B_R \times (0,\infty), \\
\nu = 0 & \text{on } \partial B_R \times (0,\infty), \\
\nu(x,0) = 0 & \text{in } B_R.
\end{cases}
\]
Set $f(x,t) = -q\bar{U}(x,t)^{2q-1}$. Since $|\nabla f|, f_t \in L^p(B_R \times (0,\tau))$ for any $p \in (1,\infty)$, $R \in (0,1)$ and $\tau > 0$, by the Sobolev inequality, $f(x,t)$ is H"older continuous on $B_R \times [1,\tau]$ for any $\tau$. Hence, by the unique solvability and the Schauder estimate, it follows that $\nu_R \in C^{2,1}(B_R \times (0,\infty))$. As a consequence, we obtain $\bar{U} \in C^{2,1}(B_1 \times (0,\infty))$, which implies that $\bar{v} \in C^{2,1}([0,1] \times (0,\infty))$. Moreover, from Lemma 4.7 and (C) in Proposition 5.1, for $\tau > 0$ there exists $c_\tau > 0$ such that
\[
|\bar{v}_r(r,t) + \bar{q}| \leq c_\tau (1-r) \quad \text{in } (0,1) \times (0,\tau).
\]
Hence we see that $\bar{v} \in C([0,1] \times (0,\infty))$. Therefore we obtain $\bar{v} \in \mathcal{K}$, which completes the proof.

6 Uniqueness of global solutions

In this section, we show the uniqueness of global solutions of (1.2) in the class $\mathcal{K}$:
\[
\mathcal{K} = \{ \bar{v} \in C^{1,0}([0,1] \times (0,\infty)); \bar{v}(r,t) > 0 \text{ in } [0,1] \times (0,\infty), \int_0^1 \int_0^\tau (\bar{v}_t^2 + \bar{v}_{rr}^2) r^{n-1} dr dt < \infty \forall \tau > 0 \}.
\]

6.1 Energy blow-up

In the previous section, we constructed a global solution satisfying (1.6). However, Theorem 1.2 does not say whether every global solution satisfies (1.6). In this subsection, we give an affirmative answer to this question.

Proposition 6.1. Let $\bar{v}(r,t)$ be any global solution of (1.2) in $\mathcal{K}$ and $T > 0$ be its vanishing time. Then $\bar{v}(r,t)$ satisfies (1.6).
Here we define the energy functional associated with (4.1).

\[ E(u) = \frac{1}{2} ||u_r||^2_2 + \frac{1}{2} ||u||^{2q}_{2q} - \frac{1}{q+1} |u(1)|^{q+1}. \]

Proposition 6.1 follows from the following result on the energy blow-up.

**Lemma 6.2.** Let \( u(r,t) \) be a positive solution of (4.1) and \( T > 0 \) be its blow-up time. Then it holds that

\[ \lim_{t \to T^-} E(u(t)) = -\infty. \]

**Proof.** Our proof is almost the same as in the proof of Theorem 3.1 in [8]. Suppose that there exists \( m > 0 \) such that \( E(u(t)) \geq -m \) for \( t \in [0,T) \). Then, from \( \partial_t E(u(t)) = -||u_t(t)||^2_2 \), we see that

\[ \int_0^t ||u_t(\tau)||^2_2 d\tau = E(u_0) - E(u(t)) \leq E(u_0) + m. \]  

(6.1)

Multiplying (4.1) by \( u_r \) and integrating over \((0,1)\), we verify that

\[ \int_0^1 u_t u_r dr = \int_0^1 \frac{n-1}{r} u_r^2 dr + \frac{1}{2} u(0,t)^{2q}. \]

Hence, by Schwarz’s inequality, there exists \( c_0 > 0 \) such that

\[ \int_0^1 u_r^2 dr \leq c_0 \int_0^1 u_t^2 dr \leq c_0 \left( \int_0^{1/2} u_t^2 dr + 2^{n-1} \int_{1/2}^1 u_r^2 r^{n-1} dr \right). \]

Therefore, by (6.1), we obtain

\[ \int_0^T dt \int_0^1 u_r^2 dr \leq c_0 \left( \int_0^T dt \int_0^{1/2} u_t^2 dr + 2^{n-1} (E(u_0) + m) \right). \]

From (b), we note that \( u_t(r,t) \) is uniformly bounded on \((0,1/2) \times (0,T)\). Thus we obtain

\[ \int_0^T dt \int_0^1 u_t^2 dr < \infty. \]  

(6.2)

Since \( u(r,t) \leq u(0,t) + \int_0^1 |u_r(r,t)| dr \), from (6.2), it holds that \( \int_0^T dt \int_0^1 u^2 dr < \infty \). Hence, from (6.1) and (6.2), we get

\[ \int_0^T dt \int_{1/2}^1 (u^2 + u_r^2 + u_t^2) dr < \infty. \]
By the Sobolev inequality, it follows that
\[
\int_0^T dt \int_{1/2}^1 u^p \, dr < \infty
\]
for any \( p \in [1, \infty) \). Then, by a parabolic regularity theory, it holds that \( u \in L^\infty((0,1) \times (0,T)) \). However this contradicts \( \lim_{t \to T} u(1,t) = \infty \), which completes the proof. \qed

**Proof of Proposition 6.1.** Let \( u_\ell(r,t) \) be a solution of
\[
\begin{cases}
  u_t = u_{rr} + \frac{n-1}{r} u_r - qu^{2q-1} & \text{in } (0,1) \times (0,\infty), \\
  \partial_r u = g_\ell(u) & \text{on } \{1\} \times (0,\infty), \\
  u(r,0) = u_0(r) & \text{in } (0,1),
\end{cases}
\]
where \( g_\ell(u) = u^q \) if \( u \leq \ell \) and \( g_\ell(u) = \ell^q \) if \( u > \ell \) and set \( v_\ell(r,t) = u_\ell(r,t)^{-q} \).

Now we claim that \( \bar{v}(r,t) \leq v_\ell(r,t) \) in \((0,1) \times (0,\infty)\) for all \( \ell \in \mathbb{N} \).

Since we can easily see that \( u_\ell(r,t) \) is uniformly bounded on \((0,1) \times (0,\infty)\), we get
\[
\inf_{(0,1) \times (0,\infty)} v_\ell(r,t) > 0.
\]

Therefore, since \( u_\ell \in C^{2,1}([0,1] \times (0,\infty)) \), it follows that \( v_\ell \in C^{2,1}([0,1] \times (0,\infty)) \).

Set \( h(r,t) = \bar{v}(r,t) - v_\ell(r,t) \), then \( \bar{v} \in \mathcal{K} \) implies that
\[
h \in C^{1,0}([0,1] \times (0,\infty)), \quad h_t, h_{rr} \in L^2_r((0,1) \times (0,\tau)) \quad \text{for } \tau > 0.
\]

Moreover \( h \) satisfies
\[
\begin{cases}
  h_t = h_{rr} + \frac{n-1}{r} h_r - \frac{\hat{q}(\bar{v} + v_\ell)}{v_\ell} h_r - \frac{\hat{q}(\bar{q}^2 - \bar{v}_r^2)}{\bar{v}v_\ell} h & \text{in } (0,1) \times (0,\infty), \\
  \partial_r h \leq 0 & \text{on } \{1\} \times (0,\infty), \\
  h(r,0) = 0 & \text{in } (0,1).
\end{cases}
\]

From the definition of \( h_+ \), we see that
\[
h_+ / \bar{v} v_\ell \leq h_+ / v_\ell^2.
\]

Hence, multiplying by \( h_+ r^{n-1} \) and integrating over \((0,1)\), we obtain
\[
\frac{1}{2} \frac{d}{dt} \| h_+ \|^2_2 \leq -\| \partial_r h_+ \|^2_2 + \hat{q} \int_0^1 \left( \frac{|\bar{v}_r + \partial_r v_\ell|}{v_\ell} |\partial_r h_+| + \frac{|\bar{q}^2 - \bar{v}_r^2|}{v_\ell^2} h_+^2 \right) r^{n-1} dr.
\]
Therefore, from $\bar{v} \in \mathcal{K}$ and (6.4), for any $\tau > 0$ there exists $c_\tau > 0$ such that for $t \in (0, \tau)$
\[
\frac{1}{2} \frac{d}{dt} \|h_+(t)\|^2 \leq c_\tau \|h_+(t)\|^2.
\]
From Gronwall's inequality, we obtain $h_+ \equiv 0$, whence the claim follows.

Repeating the same arguments above with $\bar{v}$ replaced by $v_\ell'$ ($\ell' > \ell$), we can show that $v_\ell(r, t)$ is monotone decreasing, so $u_\ell(r, t)$ is monotone increasing as $\ell \uparrow \infty$. Hence $u_\ell(r, t)$ converges to some function $u_\infty(r, t)$ for $(r, t) \in (0, 1) \times (0, \infty)$ as $\ell \to \infty$. Let $u(r, t)$ be the solution of (4.1) with the initial data $u_0$ and $T > 0$ be its blow-up time. Since $u_\ell(r, t) \leq u(r, t)$ for $t \in (0, T)$, by the parabolic regularity theory, we find that $u_\ell(r, t)$ converges to $u_\infty(r, t)$ in $C([0, 1] \times [0, T-\delta]) \cap C^{2,1}([0, 1] \times [\delta, T-\delta])$ for any small $\delta > 0$. Hence $u_\infty(r, t)$ is a solution of (4.1) in $(0, 1) \times (0, T)$ with $u_\infty(r, 0) = u_0(r)$. Then the uniqueness of solution of (4.1) assures
\[
u_\infty(r, t) \equiv u(r, t) \quad \text{in} \ (0, 1) \times (0, T).
\]
Now we claim that
\[
u_\infty(1, t) = \infty, \quad t \in (T, \infty).
\]
We define the energy functional associated with (6.3) by
\[
E_\ell(u) = \frac{1}{2} \|u_r\|^2 + \frac{1}{2} \|u\|^{2q} - G_\ell(u(1)),
\]
where $G_\ell(u) = \int_0^u g_\ell(s) ds$. Then $E_\ell(u(t))$ satisfies $\partial_t E_\ell(u(t)) = -\|\partial_t u_\ell(t)\|^2$, whence follows the monotonicity of $E_\ell(u_\ell(t))$. Hence we get
\[
-G_\ell(u_\ell(1, t)) \leq E_\ell(u_\ell(t)) \leq E_\ell(u_\ell(s)), \quad s \leq t.
\]
Since $u_\ell(s) \to u(s)$ strongly in $W^{1,2}_r(0, 1)$ as $\ell \to \infty$ for any $s < T$, we obtain
\[
\liminf_{\ell \to \infty} G_\ell(u_\ell(1, t)) \geq -E(u(s)), \quad s < T < t.
\]
From Lemma 6.2, letting $s \to T - 0$, we conclude that
\[
G(u_\infty(1, t)) \geq \liminf_{\ell \to \infty} G_\ell(u_\ell(1, t)) = \infty, \quad t > T,
\]
which implies that $u_\infty(1, t) = \infty$ for $t > T$. Hence the claim is verified. Since $\bar{v}(r, t) \leq v_\ell(r, t)$ for $\ell \in \mathbb{N}$ and $v_\ell = u_\ell^{-(q-1)}$, it follows that $\bar{v}(r, t) \leq u_\infty(r, t)^{-(q-1)}$. Thus we conclude that $\bar{v}(1, t) = 0$ for $t \in (T, \infty)$, which completes the proof. \qed
6.2 Uniqueness of global solutions

Proof of Theorem 1.4. Let \( \bar{v}_1(r, t) \) and \( \bar{v}_2(r, t) \) be two global solutions of (1.2) in \( \mathcal{K} \) with \( \bar{v}_1(r, 0) = \bar{v}_2(r, 0) \). Then we note that \( \bar{v}_1(r, t) \) coincides with \( \bar{v}_2(r, t) \) until one of them vanishes, by the uniqueness of solution of (1.2). Furthermore, by the continuity of \( \bar{v}_1 \) and \( \bar{v}_2 \), it is easy to see that \( \bar{v}_1(r, t) \) and \( \bar{v}_2(r, t) \) vanishes at the same time \( T \). Hence, by the continuity of \( \bar{v}_1 \) and \( \bar{v}_2 \), it follows that

\[ \bar{v}_1(r, T) \equiv \bar{v}_2(r, T). \]

By Proposition 6.1, we note that

\[ \bar{v}_1(1, t) = \bar{v}_2(1, t) = 0, \quad t \in [T, \infty). \]

Here we set \( h(r, t) = \bar{v}_1(r, t + T) - \bar{v}_2(r, t + T) \), then \( h(r, t) \) satisfies

\[
\begin{cases}
    h_t = h_{rr} + \frac{n - 1}{r} h_r - \frac{\hat{q}(\bar{v}_1 + \bar{v}_2)}{\bar{v}_2} h_r - \frac{\hat{q}(\bar{v}_1^2 - |\bar{v}_1 r|^2)}{\bar{v}_1 \bar{v}_2} h & \text{in } (0, 1) \times (0, \infty), \\
    h = h_r = 0 & \text{on } \{1\} \times (0, \infty), \\
    h(r, 0) = 0 & \text{in } (0, 1).
\end{cases}
\]

Multiplying (6.5) by \( h r^{n-1} \) and integrating over \((0, 1)\), we obtain

\[
\frac{1}{2} \frac{d}{dt} \| h \|_2^2 + \| h r \|_2^2 = - \int_0^1 \frac{\hat{q}(\bar{v}_1 + \bar{v}_2)}{\bar{v}_2} h h r r^{n-1} - \int_0^1 \frac{\hat{q}(|\bar{v}_1 r|^2 - |\bar{v}_1 r|^2)}{\bar{v}_1 \bar{v}_2} h^2 r^{n-1} =: -I - J.
\]

We fix \( \tau > 0 \). Here we recall that \( \bar{v}_2 \in \mathcal{K} \) and

\[ \bar{v}_2(1, t) = 0, \quad t > T, \quad \partial_r \bar{v}_2(1, t) = -\bar{q}, \quad t > 0. \]

Therefore there exist \( c_0 > 0 \) and \( r_0 \in (0, 1) \) such that \( \bar{v}_2(r, t) \geq c_0 (1 - r) \) in \((0, 1) \times (T, T + \tau)\) and \( \partial_r \bar{v}_2(r, t) \leq 0 \) in \((r_0, 1) \times (T, T + \tau)\). Then it is verified that

\[
-I = \int_0^1 \frac{2\bar{q}}{\bar{v}_2} h h r r^{n-1} - \int_0^1 \frac{\hat{q}(\bar{v}_1 r + \bar{v}_2 r + 2\bar{q})}{\bar{v}_2} h h r r^{n-1} = \int_0^{r_0} \frac{2\bar{q}}{\bar{v}_2} h h r r^{n-1} + \int_{r_0}^1 \frac{\bar{q}(\bar{v}_1 r + \bar{v}_2 r + 2\bar{q})}{\bar{v}_2} h h r r^{n-1}.
\]

Here we note that \( \lim_{r \to 1} h(r, t)^2 / \bar{v}_2(r, t + T) = 0 \). Hence, by \( \partial_r \bar{v}_2(r, t) \leq 0 \) in \((r_0, 1) \times (T, T + \tau)\), we see that

\[
\int_{r_0}^1 \frac{1}{\bar{v}_2} (h^2)_r r^{n-1} = \left[ \frac{1}{\bar{v}_2} h^2 r^{n-1} \right]_{r_0}^1 - \int_{r_0}^1 \left( \frac{r^{n-1}}{\bar{v}_2} \right)_r h^2 \leq 0, \quad t \in (0, \tau).
\]
Furthermore, we have
\[
\int_0^1 |\bar{\nu}_1 + \bar{\nu}_2 + 2\bar{q}| |h h_r| r^{n-1} \leq \frac{m + 2\bar{q}}{c_0 \sigma} \int_0^{1-\sigma} |h h_r| r^{n-1} + \frac{\mu(\sigma)}{c_0} \int_{1-\sigma}^1 |h h_r| r^{n-1} \frac{1}{1-r}
\]
for \( t \in (0, \tau) \), where
\[
m = \sup_{(r,t) \in (0,1) \times (0,T+\tau)} (|\bar{\nu}_1(r,t)| + |\bar{\nu}_2(r,t)|),
\]
\[
\mu(\sigma) = \sup_{(r,t) \in (1-\sigma,1) \times (0,T+\tau)} (|\bar{\nu}_1(r,t) + \bar{q}| + |\bar{\nu}_2(r,t) + \bar{q}|).
\]
Combining these estimates, we obtain
\[
-I \leq -I \leq 2q c_0^{-1} \int_0^{r_0} |h h_r| r^{n-1} dr + \frac{\bar{q}(m + 2\bar{q})}{c_0 \sigma} \int_0^{1-\sigma} |h h_r| r^{n-1} dr \\
+ \frac{\hat{q}\mu(\sigma)}{c_0} \int_{1-\sigma}^1 |h h_r| r^{n-1} dr.
\]
Hence there exists \( c_1 > 0 \) such that
\[
-I \leq c_1 \left( \frac{1}{\sigma} \int_0^{1} |h h_r| r^{n-1} dr + \mu(\sigma) \int_0^{1} \left( \frac{h^2}{(1-r)^2} + h_r^2 \right) r^{n-1} \right).
\]
Since \( h(1,t) = 0 \) for \( t \in (0, \tau) \), by Hardy’s inequality, we conclude that
\[
-I \leq c_2 \left( \frac{1}{\sigma} \int_0^{1} |h h_r| r^{n-1} dr + \mu(\sigma) \int_0^{1} h_r^2 r^{n-1} \right)
\]
for some \( c_2 > 0 \). In the same way, there exists \( c_3 > 0 \) such that
\[
-J \leq c_3 \left( \frac{1}{\sigma^2} \int_0^{1-\sigma} h^2 r^{n-1} dr + \frac{h^2}{(1-r)^2} \right) \\
\leq c_3 \left( \frac{1}{\sigma^2} \|h\|^2_2 + c\mu(\sigma)\|h_r\|^2_2 \right).
\]
From \( \tilde{v}_1, \tilde{v}_2 \in K \), it holds that \( \lim_{\sigma \to 0} \mu(\sigma) = 0 \). Hence we take \( \sigma > 0 \) small enough and by Schwarz’s inequality, we obtain
\[
\frac{d}{dt} \|h(t)\|^2_2 \leq c_4 \|h(t)\|^2_2, \quad t \in (0, \tau)
\]
for some \( c_4 > 0 \). Applying Gronwall’s inequality, we conclude that \( h(r,t) \equiv 0 \) in \( (0,1) \times (0, \tau) \). Since \( \tau > 0 \) is arbitrary, it holds that \( h(r,t) \equiv 0 \) on \( (0,1) \times (0, \infty) \), which completes the proof. \( \Box \)
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References


Junichi Harada  
Department of Applied Physics,  
School of Science and Engineering,  
Waseda University, 3-4-1 Okubo,  
Shinjuku-ku Tokyo, Japan 169-8555  
E-mail: harada-j@aoni.waseda.jp

Mitsuharu Ôtani  
Department of Applied Physics,  
School of Science and Engineering,  
Waseda University, 3-4-1 Okubo,  
Shinjuku-ku Tokyo, Japan 169-8555  
E-mail: otani@waseda.jp