Nonlinear periodic problems superlinear at $+\infty$ and sublinear at $-\infty$

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Abstract: We consider a nonlinear periodic problem driven by a nonlinear, nonhomogeneous differential operator with a reaction which exhibits an asymmetric growth at $+\infty$ and at $-\infty$. It is $(p-1)$-superlinear near $+\infty$ and $(p-1)$-sublinear near $-\infty$. A particular case of our problem is that of periodic equations with the scalar $p$-Laplacian and an asymmetric nonlinearity.

Using variational methods and Morse theory, we prove the existence of at least three nontrivial solutions.

Keywords: Asymmetric reaction, nonhomogeneous differential operator, C-condition, critical groups, homotopy equivalent, mountain pass theorem.

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Dedicated to the memory of Professor Francesco S. De Blasi

1 Introduction

In this paper we examine the following nonlinear periodic problem

$$\begin{align*}
- (a (|u'(t)|)) u'(t)' &= f(t, u(t)) \text{ a.e. on } T := [0, b] \\
u(0) &= u(b), \ u'(0) = u'(b). 
\end{align*}$$

(1.1)

In the above problem, the differential operator is in general nonhomogeneous and incorporates as a special case the scalar $p$-Laplacian. The reaction $f(t, x)$ is a Carathéodory function (i.e., for all $x \in \mathbb{R}$, $t \to f(t, x)$ is measurable and for a.a. $t \in T$, $x \to f(t, x)$ is continuous).

Our aim is to study the existence and multiplicity of solutions when the reaction $f(t, x)$ exhibits an asymmetric behavior near $+\infty$ and near $-\infty$ and it is $(p-1)$–superlinear in the positive direction (i.e., as $x \to +\infty$) and $(p-1)$–sublinear in the negative direction (i.e., as $x \to -\infty$).
Multiplicity results for nonlinear periodic problems driven by the scalar $p$-Laplacian were proved by Aizicovici-Papageorgiou-Staicu [1], [3], [4], Del Pino-Manasevich-Murua [8], Gasinski [11], Gasinski-Papageorgiou [13], and Yang [20].

In all the above mentioned papers, the reaction of the problem exhibits a similar growth near $+\infty$ and $-\infty$. Recently, Aizicovici-Papageorgiou-Staicu [5], [6], studied periodic eigenvalue problems driven by a nonhomogeneous differential operator.

Equations with an asymmetric reaction were studied in the context of semilinear (i.e., $p=2$) Neumann problems. We mention the works of Dong [9], de Figueiredo-Ruf [7], Perera [17] and Villegas [19]. Of these, only Perera [17] proves a multiplicity result.

Our approach uses variational methods based on critical point theory together with suitable truncation techniques and Morse theory (critical groups).

2 Mathematical Background and Hypotheses

Let $(X, \|\cdot\|)$ be a Banach space and $(X^*, \|\cdot\|_*)$ its topological dual. By $\langle \cdot, \cdot \rangle$ we denote the duality brackets for the pair $(X^*, X)$, and $\rightharpoonup$ denotes weak convergence in $X$.

Let $\varphi \in C^1(X)$. A real number $c$ is said to be a critical value of $\varphi$ if there exists $x^* \in X$ such that $\varphi'(x^*) = 0$ and $\varphi(x^*) = c$. We say that $\varphi \in C^1(X)$ satisfies the C-condition, if the following is true:

"every sequence $\{x_n\}_{n \geq 1} \subseteq X$ such that $\{\varphi(x_n)\}_{n \geq 1}$ is bounded in $\mathbb{R}$ and $$(1 + \|x_n\|) \varphi'(x_n) \to 0$$ in $X^*$ as $n \to \infty$ admits a strongly convergent subsequence."

This is in general weaker than the more common Palais-Smale condition. Nevertheless, the C-condition suffices to prove a deformation theorem and from it derive the minimax theory of certain critical values of $\varphi \in C^1(X)$. One such minimax theorem, which we recall for future use, is the so called "mountain pass theorem".

Theorem 2.1. If $\varphi \in C^1(X)$ satisfies the C-condition, $x_0, x_1 \in X$, $\rho > 0$, $\|x_1 - x_0\| > \rho$, $\max \{\varphi(x_0), \varphi(x_1)\} < \inf \{\varphi(x) : \|x - x_0\| = \rho\} = \eta_\rho$, and $c = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} \varphi(\gamma(t))$ where

$$\Gamma = \{\gamma \in C([0,1], X) : \gamma(0) = x_0, \gamma(1) = x_1\},$$

then $c \geq \eta_\rho$ and $c$ is a critical value of $\varphi$. 
Let \((Y_1, Y_2)\) be a topological pair such that \(Y_2 \subseteq Y_1 \subseteq X\). For every integer \(k \geq 0\), by \(H_k(Y_1, Y_2)\) we denote the \(k^{th}\)-relative singular homology group with integer coefficients for the pair \((Y_1, Y_2)\). Recall that \(H_k(Y_1, Y_2) = 0\) for all integers \(k < 0\).

Let \(\varphi \in C^1(X)\) and \(c \in \mathbb{R}\). We introduce the following sets:

\[
\begin{align*}
\varphi^c &= \{x \in X : \varphi(x) \leq c\}, \\
\varphi^c &= \{x \in X : \varphi(x) < c\}, \\
K_\varphi &= \{x \in X : \varphi'(x) = 0\}, \\
K_\varphi^c &= \{x \in K_\varphi : \varphi(x) = c\}.
\end{align*}
\]

The critical groups of \(\varphi\) at an isolated critical point \(x \in X\) with \(\varphi(x) = c\) (i.e., \(x \in K_\varphi^c\)) are defined by

\[
C_k(\varphi, x) = H_k(\varphi^c \cap U, (\varphi^c \cap U) \setminus \{x\}) \text{ for all } k \geq 0,
\]

where \(U\) is a neighborhood of \(x\) such that \(K_\varphi \cap \varphi^c \cap U = \{x\}\). The excision property of singular homology implies that the above definition of critical groups is independent of the particular choice of the neighborhood \(U\). Suppose that \(\varphi \in C^1(X)\) satisfies the \(C^-\)condition and \(\inf \varphi(K_\varphi) > -\infty\). Let \(c < \inf \varphi(K_\varphi)\). The critical groups of \(\varphi\) at infinity, are defined by

\[
C_k(\varphi, \infty) = H_k(X, \varphi^c) \text{ for all } k \geq 0.
\]

The second deformation theorem (see for example, Gasinski-Papageorgiou [12], p. 628) implies that the above definition of critical groups of \(\varphi\) at infinity is independent of the choice of the level \(c < \inf \varphi(K_\varphi)\).

Suppose that \(K_\varphi\) is finite. We define

\[
\begin{align*}
M(t, x) &= \sum_{k \geq 0} \text{rank } C_k(\varphi, x) t^k \text{ for all } t \in \mathbb{R}, \ x \in K_\varphi, \\
P(t, \infty) &= \sum_{k \geq 0} \text{rank } C_k(\varphi, \infty) t^k \text{ for all } t \in \mathbb{R}.
\end{align*}
\]

The Morse relation says that

\[
\sum_{x \in K_\varphi} M(t, x) = P(t, \infty) + (1 + t) Q(t) \tag{2.1}
\]

where \(Q(t) = \sum_{k \geq 0} \beta_k t^k\) is a formal series in \(t \in \mathbb{R}\) with nonnegative integer coefficients.
In the analysis of problem (1.1), we will use the Sobolev space
\[ W_{\text{per}}^{1,p}(T) = \{ u \in W^{1,p}(T) : u(0) = u(b) \}, \]
with \(1 < p < \infty\). The space \(W_{\text{per}}^{1,p}(T)\) is embedded compactly into \(C(T)\), and so, the evaluations at \(t = 0\) and \(t = b\) make sense.

In the sequel, for notational economy we set
\[ W := W_{\text{per}}^{1,p}(T). \]

In addition to the Sobolev space \(W\) we will also use the Banach space
\[ \widehat{C}^1(T) = C^1(T) \cap W. \]

This is an ordered Banach space with positive cone
\[ \widehat{C}_+ = \{ u \in \widehat{C}^1(T) : u(t) \geq 0 \text{ for all } t \in T \}. \]

This cone has a nonempty interior, given by
\[ \text{int } \widehat{C}_+ = \{ u \in \widehat{C}_+ : u(t) > 0 \text{ for all } t \in T \}. \]

Throughout this paper, the norm of the Banach space \(W\) will be denoted by \(\| . \|\), i.e.,
\[ \| u \| = \left( \| u \|_p^p + \| u' \|_p^p \right)^{\frac{1}{p}} \text{ for all } u \in W, \]
with \(\| . \|_p\) being the norm of \(L^p(T)\).

Given \(x \in \mathbb{R}\), we set \(x^\pm = \max\{\pm x, 0\}\). We have
\[ x = x^+ - x^-, \text{ and } |x| = x^+ + x^- . \]

Then for every \(u \in W\), we define \(u^\pm(\cdot) = u(\cdot)^\pm\) and we have
\[ u = u^+ - u^-, \quad |u| = u^+ + u^- \text{ and } u^\pm \in W. \]

Also, if \(h : T \times \mathbb{R} \rightarrow \mathbb{R}\) is a measurable function, then we define
\[ N_h(u)(\cdot) = h(\cdot, u(\cdot)) \text{ for all } u \in W \]
(the Nemytskii map corresponding to \(h(t, x)\)).

Finally, by \(\| . \|_1\) we denote the Lebesgue measure on \(\mathbb{R}\).

Our hypotheses on the map \(a\) in problem (1.1) are the following:
**H** (a): $a : (0, \infty) \to (0, \infty)$ is a $C^1$-function such that:

(i) $x \to a (x) x$ is strictly increasing on $(0, \infty)$, $a (x) x \to 0$ as $x \to 0^+$ and 

$$\frac{a'(x) x}{a(x)} \to C > -1 \text{ as } x \to 0^+;$$

(ii) there exists $\tilde{C} > 0$ and $1 < p < \infty$ such that 

$$|a (x) x| \leq \tilde{C} \left(1 + |x|^{p-1}\right) \text{ for all } x \in \mathbb{R};$$

(iii) there exists $C_0 > 0$ such that 

$$a'(x) x^2 \geq C_0 x^{p-1} \text{ for all } x > 0;$$

(iv) if $G_0 (x) = \int_0^x a(s) \, ds$ for all $x \geq 0$, then 

$$pG_0 (x) - a(x) x^2 \geq 0 \text{ for all } x > 0.$$

Evidently $G_0 (.)$ is strictly convex and strictly increasing on $(0, \infty)$. We set 

$$G(x) = G_0 (|x|) \text{ for all } x \in \mathbb{R}.$$ 

Then $G(.)$ is strictly convex and for $x \neq 0$ we have 

$$G'(x) = G'_0 (|x|) \frac{x}{|x|} = a(|x|) x.$$ 

So $G(.)$ is the primitive of the function $x \to a (|x|) x$, $x \in \mathbb{R}$. Since $G_0 (.)$ is convex and $G_0 (0) = 0$, we have 

$$G_0 (x) \leq a(x) x^2 \text{ for all } x > 0. \quad (2.2)$$ 

Using (2.1) and hypotheses **H** (a) (ii), (iii), we obtain 

$$\frac{C_0}{p} |x|^p \leq G(x) \leq C_1 (1 + |x|^p) \text{ for all } x \in \mathbb{R} \text{ and some } C_1 > 0. \quad (2.3)$$
Examples: The following functions satisfy the above hypotheses:

\[
\begin{align*}
  a_1(x) &= |x|^{p-2}x \text{ with } 1 < p < \infty, \\
a_2(x) &= |x|^{p-2}x + |x|^{q-2}x \text{ with } 1 < q < p < \infty, \\
a_3(x) &= (1 + x^2)^{\frac{p-2}{2}}x \text{ with } 1 < p < \infty, \\
a_4(x) &= |x|^{p-2}x + \frac{|x|^{p-2}x}{1 + |x|^p} \text{ with } 1 < p < \infty.
\end{align*}
\]

The corresponding potential (primitive) functions are:

\[
\begin{align*}
  G_1(x) &= \frac{1}{p} |x|^p, \\
  G_2(x) &= \frac{1}{p} |x|^p + \frac{1}{q} |x|^q, \\
  G_3(x) &= \frac{1}{p} \left[ (1 + x^2)^{\frac{q}{2}} - 1 \right], \\
  G_4(x) &= \frac{1}{p} |x|^p + \ln (1 + |x|^p).
\end{align*}
\]

Note that \( a_1(x) \) corresponds to the scalar \( p \)-Laplacian, \( a_2(x) \) corresponds to the scalar \((p,q)\)-Laplacian, and \( a_3(x) \) corresponds to the generalized scalar \( p \)-mean curvature operator.

Let \( A : W \to W^* \) be the nonlinear map defined by

\[
\langle A(u), v \rangle = \int_0^b a \left( |u'(t)| \right) u'(t) v'(t) \, dt \text{ for all } u, v \in W. \tag{2.4}
\]

The following result can be found in Papageorgiou-Rocha-Staicu [16].

**Proposition 2.2.** If hypotheses \( \mathbf{H}(a) \) hold, then the nonlinear operator \( A : W \to W^* \) defined by (2.4) is bounded (i.e., it maps bounded sets to bounded sets), continuous, strictly monotone (hence maximal monotone, too), and of type \((S)_+\), i.e., if \( u_n \wto u \) in \( W \) and

\[
\limsup_{n \to \infty} \langle A(u_n), u_n - u \rangle \leq 0,
\]

then \( u_n \to u \) in \( W_0^{1,p}(\Omega) \).

Let \( f_0 : T \times \mathbb{R} \to \mathbb{R} \) be a Carathéodory function such that

\[
|f_0(t,x)| \leq a(t) \left( 1 + |x|^{r-1} \right) \text{ for a.a. } t \in T, \text{ all } x \in \mathbb{R},
\]
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with $a \in L^1(T)_+$ and $1 < r < \infty$. We set

$$F_0(t, x) = \int_0^x f_0(t, s) \, ds$$

and consider the $C^1$–functional $\psi_0 : W \to \mathbb{R}$ defined by

$$\psi_0(u) = \int_0^b G(u'(t)) \, dt - \int_0^b F_0(t, u(t)) \, dt$$

for all $u \in W$.

The following result can be found in Aizicovici-Papageorgiou-Staicu [6]).

**Proposition 2.3.** If hypotheses $H(a)$ hold and $u_0 \in W$ is a local $\widehat{C}^1(T)$-minimizer of $\psi_0$ (i.e., there exists $\rho_0 > 0$ such that $\psi_0(u_0) \leq \psi_0(u_0 + h)$ for all $h \in \widehat{C}^1(T)$ with $\|h\|_{\widehat{C}^1(T)} \leq \rho_0$), then $u_0 \in \widehat{C}^1(T)$ and it is a local $W$-minimizer of $\psi_0$, (i.e., there exists $\rho_1 > 0$ such that $\psi_0(u_0) \leq \psi_0(u_0 + h)$ for all $h \in W$ with $\|h\| \leq \rho_1$).

The hypotheses on the reaction $f(t, x)$ are the following:

$H(f)$: $f : T \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that $f(t, 0) = 0$ a.e. on $T$ and

1. there exist $a \in L^1(T)_+$ and $p < r < \infty$ such that
   $$|f(t, x)| \leq a(t) \left(1 + |x|^{r-1}\right)$$
   for a.a. $t \in T$, all $x \in \mathbb{R}$;

2. $\lim_{x \to +\infty} \frac{F(t, x)}{x^p} = +\infty$ uniformly for a.a. $t \in T$, and there exist $\tau > r - p$ and $\beta_0 > 0$ such that
   $$\beta_0 \leq \liminf_{x \to +\infty} \frac{f(t, x) - pF(t, x)}{x^{\tau}}$$
   uniformly for a.a. $t \in T$,

where

$$F(t, x) := \int_0^x f(t, s) \, ds.$$
(iii) there exist functions \( \hat{\theta}, \theta \in L^\infty (T) \) such that:

\[
\hat{\theta} (t) \leq \theta (t) \leq 0 \quad \text{for a.a.} \ t \in T, \ \theta \neq 0,
\]

\[
\hat{\theta} (t) \leq \lim \inf_{x \to -\infty} \frac{pF(t,x)}{|x|^p} \leq \lim \sup_{x \to +\infty} \frac{pF(t,x)}{|x|^p} \leq \theta (t)
\]

uniformly for a.a. \( t \in T \),

\[
\lim \sup_{x \to -\infty} [pF(t,x) - f(t,x)x] < +\infty
\]

uniformly for a.a. \( t \in T \);

(iv) there exist constants \( \tilde{\xi}_0, \delta_0 > 0 \) such that

\[
F(t,x) \leq 0 \quad \text{for a.a.} \ t \in T, \ \text{all} \ |x| \leq \delta_0, \ F \left( t, -\tilde{\xi}_0 \right) dt < 0
\]

and for every \( \rho > 0 \), there exists \( \hat{\xi}_\rho > 0 \) such that for a.a. \( t \in T \),

\[
x \to f(t,x) + \hat{\xi}_\rho |x|^{p-2} x
\]

is nondecreasing on \( [-\rho, \rho] \).

Remarks: Hypotheses \( H(f)(ii), (iii) \) reveal the asymmetric character of the nonlinearity \( f(t,.) \). By virtue of hypothesis \( H(f)(ii) \), near \( +\infty, x \to f(t,x) \) is \( (p-1) \)–superlinear. However, note that here we do not use the usual in such cases Ambrosetti-Rabinowitz condition (unilateral version). Instead, we employ a weaker requirement.

Hypothesis \( H(f)(iii) \) implies that for a.a. \( t \in T, \) near \( -\infty, x \to f(t,x) \) is \( (p-1) \)–sublinear. So, we have, a different growth for \( f(t,.) \) in the positive and negative direction, respectively.

3 Three Solutions Theorem

In this section, we establish the existence of three nontrivial solutions for problem (1.1). To this end, let \( \varphi : W \to \mathbb{R} \) be the energy functional for problem (1.1) defined by

\[
\varphi(u) = \int_0^b G(u'(t)) \, dt - \int_0^b F(t,u(t)) \, dt, \ \text{for all} \ u \in W.
\]
Evidently \( \varphi \in C^1(W) \). Also, we consider the following perturbations-truncations of \( f(t,.) \):

\[
\begin{align*}
\hat{f}_+(t,x) &= \begin{cases} 
0 & \text{if } x \leq 0, \\
 f(t,x) + x^{p-1} & \text{if } x > 0,
\end{cases} \\
\hat{f}_-(t,x) &= \begin{cases} 
 f(t,x) + |x|^{p-2} x & \text{if } x < 0 \\
0 & \text{if } x \geq 0.
\end{cases}
\end{align*}
\]  

Both are Carathéodory functions. We set

\[
\hat{F}_\pm(t,x) = \int_0^x \hat{f}_\pm(t,s) \, ds
\]

and introduce the \( C^1 \)-functionals \( \hat{\varphi}_\pm : W \to \mathbb{R} \) by

\[
\hat{\varphi}_\pm(u) = \int_0^b G(u'(t)) \, dt + \frac{1}{p} \|u\|_p^p - \int_0^b \hat{F}_\pm(t,u(t)) \, dt, \text{ for all } u \in W.
\]

**Proposition 3.1.** If hypotheses \( H(a) \) and \( H(f) \) hold, then the functional \( \varphi \) satisfies the \( C^- \) condition.

**Proof.** Let \( \{u_n\}_{n \geq 1} \) be a sequence in \( W \) such that

\[
|\varphi(u_n)| \leq M_1 \text{ for some } M_1 > 0, \text{ all } n \geq 1
\]  

and

\[
(1 + \|u_n\|) \varphi'(u_n) \to 0 \text{ in } W^* \text{ as } n \to \infty.
\]  

From (3.3), we have

\[
\left| \langle A(u_n), h \rangle - \int_0^b f(t,u_n) \, h \, dt \right| \leq \frac{\varepsilon_n \|h\|}{1 + \|u_n\|} \text{ for all } h \in W,
\]

with \( \varepsilon_n \to 0^+ \). In (3.4), we choose \( h = u_n^+ \in W \). Then

\[
- \int_0^b a \left( |(u_n^+)'| \right) \left( (u_n^+)' \right)^2 dt + \int_0^b f(t,u_n^+) \, u_n^+ \, dt \leq \varepsilon_n \text{ for all } n \geq 1.
\]

(3.5)
From (3.2) we have
\[
\begin{align*}
\int_0^b pG \left( (u_n^+) \right) dt &+ \int_0^b pG \left( -(u_n^-) \right) dt \\
\int_0^b pF \left( (u_n^+) \right) dt &- \int_0^b pF \left( t, -(u_n^-) \right) dt \\
\leq pM_1 & \text{ for all } n \geq 1.
\end{align*}
\] (3.6)

Adding (3.5) and (3.6), we obtain
\[
\begin{align*}
\int_0^b \left[ pG \left( (u_n^+) \right) - a \left( \left| (u_n^+) \right| \right) \left( (u_n^+) \right)^2 \right] dt \\
+ \int_0^b \left[ f \left( t, u_n^+ \right) u_n^+ - pF \left( t, u_n^+ \right) \right] dt \\
+ \int_0^b pG \left( -(u_n^-) \right) dt &- \int_0^b pF \left( t, -u_n^- \right) dt \\
\leq pM_1 & \text{ for all } n \geq 1.
\end{align*}
\] (3.7)

By virtue of hypothesis \(\mathbf{H}(f)\) \((iii)\), given \(\varepsilon > 0\), we can find \(a_\varepsilon \in L^1 (T)\) such that
\[
F \left( t, x \right) \leq \frac{1}{p} \left( \theta \left( t \right) \varepsilon \right) |x|^p + a_\varepsilon \left( t \right) \text{ for a.a. } t \in T, \text{ all } x \leq 0.
\] (3.8)

Using (3.8) in (3.7), we obtain
\[
\begin{align*}
\int_0^b \left[ pG \left( (u_n^+) \right) - a \left( \left| (u_n^+) \right| \right) \left( (u_n^+) \right)^2 \right] dt \\
+ \int_0^b \left[ f \left( t, u_n^+ \right) u_n^+ - pF \left( t, u_n^+ \right) \right] dt \\
+ C_0 \left\| (u_n^-) \right\|^p \left[ - \int_0^b \theta \left( t \right) u_n^- \right|^{p} dt - \varepsilon \left\| u_n^- \right\|^p - C_2 \\
\leq pM_1 & \text{ for some } C_2 > 0, \text{ all } n \geq 1,
\end{align*}
\]
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(see (3.8) and (2.3)), hence

\[
\int_0^b \left[ pG\left( \left( u^+_n \right) \right) - a\left( \left( \left( u^+_n \right) \right) \right) \left( \left( u^+_n \right) \right)^2 \right] dt
\]

\[
+ \int_0^b \left[ f\left( t, u^+_n \right) u^+_n - pF\left( t, u^+_n \right) \right] dt + \xi_0 \left\| u^-_n \right\|^p
\]

\[
\leq pM_1 + C_2 =: C_3 \text{ for all } n \geq 1, \text{ some } \xi_0 > 0
\]

(see Aizicovici-Papageorgiou-Staicu [6], Lemma 2.1), therefore

\[
\int_0^b \left[ f\left( t, u^+_n \right) u^+_n - pF\left( t, u^+_n \right) \right] dt \leq C_3 \text{ for all } n \geq 1 \text{ (see } H(a) (iv) \).
\]  

Hypotheses $H(f)(i), (ii)$ imply that we can find $\beta_1 \in (0, \beta_0)$ and $a_1 \in L^1(T)_+$ such that

\[
\beta_1 x^r - a_1(t) \leq f(t, x) x - pF(t, x) \text{ for a.a. } t \in T, \text{ all } x \geq 0.
\]

(3.10)

Returning to (3.9) and using (3.10), we obtain

\[
\beta_1 \left\| u^+_n \right\| \leq M_2 \text{ for some } M_2 > 0, \text{ all } n \geq 1,
\]

hence

\[
\left\{ u^+_n \right\}_{n \geq 1} \subset L^r(T) \text{ is bounded.}
\]

(3.11)

In (3.4) we choose $h = u^+_n \in W$ and use hypothesis $H(a)(iii)$ to arrive at

\[
C_0 \left\| \left( u^+_n \right) \right\|^p - \int_0^b f\left( t, u^+_n \right) u^+_n dt \leq \varepsilon_n \text{ for all } n \geq 1, \text{ (see (2.3) ).}
\]

(3.12)

By virtue of $H(f)(i)$ we have

\[
f\left( t, u^+_n(t) \right) u^+_n(t) \leq a(t) \left( u^+_n(t) + u^+_n(t) \right) \text{ for a.a. } t \in T, \text{ all } n \geq 1.
\]

(3.13)

Using (3.13) in (3.12) we obtain

\[
\left\| \left( u^+_n \right) \right\|^p \leq \varepsilon_n + C_5 \left( \left\| u^+_n \right\| + \left\| u^+_n \right\|^r \right) \text{ for some } C_5 > 0, \text{ all } n \geq 1.
\]

(3.14)
From hypothesis $\textbf{H}(f) (ii)$ it is clear that we can always assume that $\tau \leq r < \infty$. So, we can find $t \in [0, 1)$ such that

$$\frac{1}{r} = \frac{1 - t}{\tau}.$$  \hspace{1cm} (3.15)

Invoking the interpolation inequality (see, for example, Gasinski-Papageorgiou [12], p. 905), we have

$$\|u_n^+\|_r \leq \|u_n^+\|_\tau^{1-t} \|u_n^+\|_\infty^t$$  for all $n \geq 1,$

hence

$$\|u_n^+\|_r \leq C_6 \|u_n^+\|^{tr}$$  for some $C_6 > 0$, all $n \geq 1$ \hspace{1cm} (3.16)

(see (3.11)). Using (3.16) in (3.14) we have

$$\left\| (u_n^+)' \right\|_p^p \leq C_7 \left( 1 + \|u_n^+\| + \|u_n^+\|^{tr} \right)$$  for some $C_7 > 0$, all $n \geq 1,$

therefore

$$\|u_n^+\|_p^p \leq C_8 \left( 1 + \|u_n^+\| + \|u_n^+\|^{tr} \right)$$  for some $C_8 > 0$, all $n \geq 1$ \hspace{1cm} (3.17)

(see (3.11) and Gasinski-Papageorgiou [12], p. 227)). From (3.15) we have

$$tr = r - \tau < p$$  (see hypothesis $\textbf{H}(f) (ii)$).

So, from (3.17) it follows that

$$\{ u_n^+ \}_{n \geq 1} \subset W$$  is bounded.  \hspace{1cm} (3.18)

Then from (3.7), (3.18) and hypotheses $\textbf{H}(a) (iv)$, $\textbf{H}(f) (i)$, we have

$$\int_0^b pG \left( - (u_n^-)' \right) dt - \int_0^b F \left( t, -u_n^- \right) dt \leq M_3$$  for some $M_3 > 0$, all $n \geq 1.$

Using (3.18) and Lemma 2.1 of Aizicovici-Papageorgiou-Staicu [6], we have

$$\xi_0 \|u_n^-\|_p^p \leq M_4$$  for some $M_4$, $\xi_0 > 0$, all $n \geq 1,$

hence

$$\{ u_n^- \}_{n \geq 1} \subset W$$  is bounded.  \hspace{1cm} (3.19)

From (3.18) and (3.19) it follows that $\{ u_n \}_{n \geq 1} \subset W$ is bounded and so, we may assume that

$$u_n \overset{w}{\rightarrow} u \text{ in } W \text{ and } u_n \rightarrow u \text{ in } C(T).$$  \hspace{1cm} (3.20)
In (3.4) we choose $h = u_n - u \in W$, pass to the limit as $n \to \infty$ and use (3.20). Then
\[
\lim_{n \to \infty} \langle A(u_n), u_n - u \rangle = 0,
\]
therefore
\[
u_n \to u \text{ in } W
\]
(see Proposition 2.2). This proves that the functional $\varphi$ satisfies the $C$–condition.

**Proposition 3.2.** If hypotheses $H(a)$ and $H(f)$ hold, then the functional $\hat{\varphi}_+$ satisfies the $C$–condition.

**Proof.** Let $\{u_n\}_{n \geq 1}$ be a sequence in $W$ such that
\[
|\hat{\varphi}_+(u_n)| \leq M_5 \text{ for some } M_5 > 0, \text{ all } n \geq 1. \tag{3.21}
\]
and
\[
(1 + \|u_n\|) \hat{\varphi}_+'(u_n) \to 0 \text{ in } W^* \text{ as } n \to \infty. \tag{3.22}
\]
From (3.22) we have
\[
\left| \langle A(u_n), h \rangle + \int_0^b |u_n|^{p-2} u_n h dt - \int_0^b \hat{f}_+(t, u_n) h dt \right|
\leq \frac{\varepsilon_n \|h\|}{1 + \|u_n\|} \text{ for all } h \in W, \text{ with } \varepsilon_n \to 0^+.
\tag{3.23}
\]
In (3.23), we choose $h = -u_n^- \in W$. Then from hypothesis $H(a)(iii)$ and (3.1), we have
\[
C_0 \left\| \left( u_n^- \right)' \right\|^p_p + \|u_n^-\|^p_p \leq \varepsilon_n \text{ for all } n \geq 1
\]
and hence
\[
u_n^- \to 0 \text{ in } W \text{ as } n \to \infty. \tag{3.24}
\]
From (3.21) and (3.24) we have
\[
\int_0^b pG \left( (u_n^+)' \right) dt + \|u_n^+\|^p_p - \int_0^b p\hat{F}(t, u_n^+) dt \leq M_6
\tag{3.25}
\]
for some $M_6 > 0$, all $n \geq 1$.\[\square\]
Also, if in (3.23), we choose \( h = u_n^+ \in W \), then
\[
- \int_0^b a \left( \left| \left( u_n^+ \right)' \right| \right) \left( \left( u_n^+ \right)' \right)^2 dt - \| u_n^+ \|^p_p + \int_0^b f_\pm (t, u_n^+) \ dt \leq \varepsilon_n \text{ for all } n \geq 1. \tag{3.26}
\]

We add (3.25) and (3.26). Using hypothesis \( H(a)(iv) \) and (3.1), we obtain
\[
\int_0^b \left[ f(t, u_n^+) \ u_n^+ - pF(t, u_n^+) \right] \ dt \leq M_7 \text{ for some } M_7 > 0, \text{ all } n \geq 1. \tag{3.27}
\]

Using (3.10), we infer that \( \{ u_n^+ \}_{n \geq 1} \subset L^r(T) \) is bounded. By virtue of hypothesis \( H(f)(i) \), we have
\[
|f(t, x) x| \leq a(t) \left| |x| + |x|^r \right| \text{ for a.a. } t \in T, \text{ all } x \in \mathbb{R}. \tag{3.28}
\]

In (3.23), we choose \( h = u_n^+ \in W \) and obtain
\[
C_0 \left\| (u_n^+)' \right\|^p + \| u_n^+ \|^p_p \leq \varepsilon_n + \int_0^b f(t, u_n^+) \ u_n^+ \ dt
\]
\[
\leq C_9 \left( 1 + \| u_n^+ \|^r_p \right) \text{ for some } C_9 > 0, \text{ all } n \geq 1
\]
(see (3.26) and (2.3)). Using (3.15), the interpolation inequality and the boundedness of \( \{ u_n^+ \}_{n \geq 1} \subset L^r(T) \), as in the proof of Proposition 3.1 (see (3.16) and (3.17)), we obtain
\[
\| u_n^+ \|^r_p \leq C_{10} \left( 1 + \| u_n^+ \|^r_p \right) \text{ for some } C_{10} > 0, \text{ all } n \geq 1,
\]

hence
\[
\{ u_n^+ \}_{n \geq 1} \subset W \text{ is bounded} \tag{3.29}
\]
(since \( tr = r - \tau < p \), see \( H(f)(ii) \)). From (3.24) and (3.29) it follows that \( \{ u_n \}_{n \geq 1} \subset W \) is bounded. So, we may assume that
\[
u_n \rightarrow u \text{ in } W \text{ and } u_n \rightarrow u \text{ in } C(T). \tag{3.30}
\]

In (3.23) we choose \( h = u_n - u \in W \), pass to the limit as \( n \to \infty \) and use (3.30). Then
\[
\lim_{n \to \infty} \langle A(u_n), u_n - u \rangle = 0,
\]
therefore
\[
u_n \rightarrow u \text{ in } W
\]
(see Proposition 2.2). This proves that the functional \( \mathring{\varphi}_+ \) satisfies the \( C \)-condition. \( \square \)
Proposition 3.3. If hypotheses $H(a)$ and $H(f)$ hold, then the functional $\hat{\varphi}_-$ is coercive.

Proof. By virtue of hypotheses $H(f)(i)$ and $(iii)$, given $\varepsilon > 0$ we can find $a_2 \in L^1(T)$ such that

$$F(t,x) \leq \frac{1}{p} (\theta(t) + \varepsilon) |x|^p + a_2(t) \text{ for a.a. } t \in T, \text{ all } x \leq 0. \quad (3.31)$$

Then for all $u \in W$, we have

$$\hat{\varphi}_-(u) = \int_0^b G(u'(t)) \, dt + \frac{1}{p} \|u\|_p^p - \int_0^b \hat{F}_-(t,u(t)) \, dt$$

$$\geq \frac{C_0}{p} \|u'\|^p_p - \frac{1}{p} \int_0^b \theta(t) |u(t)|^p \, dt - \frac{\varepsilon}{p} \|u\|^p - \|a_2\|_1$$

(see (2.2) and (3.31))

$$\geq \frac{\xi_0 - \varepsilon}{p} \|u\|^p - \|a_2\|_1 \text{ with } \xi_0 > 0$$

(see [6]), Lemma 2.1).

Choosing $\varepsilon \in (0, \xi_0)$ in the last inequality, we conclude that $\hat{\varphi}_-$ is coercive. \qed

Now we are ready to produce two constant sign solutions.

Proposition 3.4. If hypotheses $H(a)$ and $H(f)$ hold, then problem (1.1) has at least two constant sign solutions $u_0 \in \text{int } \widehat{\mathcal{C}}_+$ and $v_0 \in -\text{int } \widehat{\mathcal{C}}_+$. Moreover, $v_0$ is a local minimizer of $\varphi$.

Proof. First we show that $u = 0$ is a local minimizer of the functional $\hat{\varphi}_+$. So, let $u \in \widehat{\mathcal{C}}^1(T)$ with $\|u\|_{\widehat{\mathcal{C}}^1(T)} \leq \delta_0$, where $\delta_0 > 0$ is as postulated by hypothesis $H(f)(iv)$. Then

$$\hat{\varphi}_+(u) = \int_0^b G(u'(t)) \, dt + \frac{1}{p} \|u\|_p^p - \int_0^b \hat{F}_+(t,u(t)) \, dt$$

$$\geq \frac{C_0}{p} \|u'\|^p_p \text{ (see (2.2) and } H(f)(iv)),}$$
hence \( u = 0 \) is a local \( \hat{C}^1 (T) \)–minimizer of the functional \( \hat{\varphi}_+ \), therefore \( u = 0 \) is a local \( W \)–minimizer of the functional \( \hat{\varphi}_+ \) (see Proposition 2.3). This implies that we can find \( \rho \in (0, 1) \) small, such that

\[
\hat{\varphi}_+(0) = 0 < \inf \{ \hat{\varphi}_+(u) : \|u\| = \rho \} =: \hat{\eta}_p^+ \tag{3.32}
\]

(see Aizicovici-Papageorgiou-Staicu [2], p. 57). For \( \xi \in (0, \infty) \) we have

\[
\hat{\varphi}_+(\xi) = -\int_0^b F_+(t, \xi) \, dt \quad \text{(see (3.1))},
\]

hence

\[
\hat{\varphi}_+(\xi) \to -\infty \text{ as } \xi \to +\infty \quad \text{(see } H(f) \text{ (ii) )}. \tag{3.33}
\]

From Proposition 3.2 we know that \( \hat{\varphi}_+ \) satisfies the \( C \)–condition. This fact together with (3.32) and (3.33) permit the use of Theorem 2.1 (the mountain pass theorem). So, we can find \( u_0 \in W \) such that

\[
\hat{\varphi}_+(0) = 0 < \hat{\eta}_p^+ \leq \hat{\varphi}_+(u_0) \tag{3.34}
\]

and

\[
\hat{\varphi}_+'(u_0) = 0. \tag{3.35}
\]

From (3.34) we see that \( u_0 \neq 0 \). From (3.35) we have

\[
A(u_0) + |u_0|^{p-2} u_0 = N_{\hat{f}_+}(u_0). \tag{3.36}
\]

On (3.36) we act with \(-u_0^- \in W\), and use (3.1) and Proposition 3.2 to obtain

\[
C_0 \left\| (u_0^-)' \right\|^p_p + \left\| u_0^- \right\|^p_p \leq 0,
\]

therefore

\[
u_0 \geq 0, \quad u_0 \neq 0.
\]

Hence, (3.36) becomes

\[
A(u_0) = N_f(u_0) \quad \text{(see (3.1))},
\]

and we get

\[
\begin{cases}
-(a(|u_0'(t)|)u_0'(t))' = f(t, u_0(t)) \quad \text{a.e. on } T, \\
u_0(0) = u_0(b), u_0'(0) = u_0'(b) \cdot \tag{3.37}
\end{cases}
\]
Periodic problems superlinear at $+\infty$ and sublinear at $-\infty$.

Then $u_0 \in \hat{C}^1(T)$. Let $\rho = \|u_0\|_\infty$ and $\xi_\rho > 0$ be as postulated by hypothesis $H (f) (iv)$. From (3.37) we have

$$- (a (|u'_0(t)|) u'_0(t))^t + \xi_\rho u_0(t)^{p-1} = f(t, u_0(t)) + \xi_\rho u_0(t)^{p-1} \geq 0 \text{ a.e. on } T,$$

hence

$$(a (|u'_0(t)|) u'_0(t))^t \leq \xi_\rho u_0(t)^{p-1} \text{ a.e. on } T.$$ (3.38)

From (3.38) and the strong maximum principle of Pucci-Serrin ([18], p.111) it follows that $u_0(t) > 0$ for all $t \in (0, b)$.

Then the boundary point theorem of Pucci-Serrin ([18], p.120) implies that $u_0 \in int C_+$. From Proposition 3.3 we know that $\hat{\varphi}_-$ is coercive. Also, using the Sobolev embedding theorem, we see that $\hat{\varphi}_-$ is sequentially lower semicontinuous.

So, by the Weierstrass theorem we can find $v_0 \in W$ such that

$$\hat{\varphi}_-(v_0) = \inf \{ \hat{\varphi}_-(u) : u \in W \}. \quad (3.39)$$

Let $v = -\tilde{\xi}_0 \in -int \hat{C}_+$ be as in hypothesis $H (f) (iv)$. We have

$$\hat{\varphi}_-(v_0) = -\int_0^b F(t, -\tilde{\xi}_0) dt < 0$$

(see (3.1) and hypothesis $H (f) (iv)$), hence

$$\hat{\varphi}_-(v_0) < 0 = \hat{\varphi}_-(0)$$

(see (3.39)), therefore

$$v_0 \neq 0.$$

From (3.39) we have

$$\hat{\varphi}'_-(v_0) = 0.$$

which implies

$$A(v_0) + |v_0|^{p-2} v_0 = N_{\hat{f}_-}(v_0). \quad (3.40)$$

Acting on (3.40) with $v_0^+ \in W$ and using (3.1) and Proposition 3.2, we obtain

$$C_0 \left\| (v_0^+)' \right\|^p + \left\| v_0^+ \right\|^p \leq 0,$$

hence

$$v_0 \leq 0, \ v_0 \neq 0.$$
Then (3.40) becomes
\[ A(v_0) = N_f(v_0), \]
and we get
\[
\begin{cases}
-(a(|v'_0(t)|)v'_0(t))' = f(t, v_0(t)) \text{ a.e. on } T, \\
v_0(0) = v_0(b), v'_0(0) = v'_0(b).
\end{cases}
\]
Hence \( v_0 \in \widehat{C}^1(T) \), and as before, using the results of Pucci-Serrin ([18], pp. 111, 120), we obtain \( v_0 \in -\text{int } \widehat{C}_+ \). Note that
\[ \varphi |_{-\widehat{C}_+} = \widehat{\varphi} |_{-\widehat{C}_+}. \]
Therefore \( v_0 \in -\text{int } \widehat{C}_+ \) is a local \( \widehat{C}(T) \)-minimizer of \( \varphi \), and from Proposition 2.3 it follows that \( v_0 \in -\text{int } \widehat{C}_+ \) is a local \( W \)-minimizer of \( \varphi \).

Next, using Morse theory, we will produce a third nontrivial solution for problem (1.1). To this end, we start by computing the critical groups of \( \varphi \) at infinity.

**Proposition 3.5.** If hypotheses \( H(a) \) and \( H(f) \) hold, then \( C_k(\varphi, \infty) = 0 \) for all \( k \geq 0 \).

**Proof.** Let \( \psi := \varphi |_{\widehat{C}^1(T)} \). The regularity properties of solutions of (1.1) imply that
\[ K_\psi = K_\varphi = K. \]
Since \( \widehat{C}^1(T) \hookrightarrow W \) densely, from Palais [15], we have
\[ H_k(W, \varphi^\alpha) = H_k(\widehat{C}^1(T), \varphi^\alpha) \text{ for all } \alpha \in \mathbb{R}, \text{ all } k \geq 0. \tag{3.41} \]
Let \( \alpha < \inf \varphi(K) = \inf \psi(K) \). We have
\[ C_k(\varphi, \infty) = H_k(W, \varphi^\alpha) = H_k(W, \varphi^\alpha) \text{ for all } k \geq 0, \tag{3.42} \]
\[ C_k(\psi, \infty) = H_k(\widehat{C}^1(T), \psi^\alpha) = H_k(\widehat{C}^1(T), \psi^\alpha) \text{ for all } k \geq 0, \tag{3.43} \]
(see Granas-Dugundji [14], p.407). Then from (3.41), (3.42), (3.43), it follows that in order to prove the proposition, it suffices to show that
\[ H_k(\widehat{C}^1(T), \psi^\alpha) = 0 \text{ for all } k \geq 0, \alpha < 0 \text{ with } |\alpha| \text{ big.} \]
To this end, we introduce the following sets

$$\partial B^C_1 = \left\{ u \in \hat{C}^1 (T) : \|u\|_{\hat{C}^1(T)} = 1 \right\},$$

$$\partial B^C_{1,+} = \{ u \in \partial B^C_1 : u(t) > 0 \text{ for some } t \in (0,b) \}.$$ 

Let \( h_+: [0,1] \times \partial B^C_{1,+} \to \partial B^C_{1,+} \) be the homotopy defined by

$$h_+(t,u) = \frac{(1-t)u + t\hat{u}_0}{\| (1-t)u + t\hat{u}_0 \|_{\hat{C}^1(T)}} \text{ for all } (t,u) \in [0,1] \times \partial B^C_{1,+},$$

where \( \hat{u}_0 \in \text{int } \hat{C}_+ \) with \( \| \hat{u}_0 \|_{\hat{C}^1(T)} = 1 \). We have

$$h_+(0,u) = u, \ h_+(1,u) = \hat{u}_0,$$

hence \( \partial B^C_{1,+} \) is contractible in itself. For \( u \in \partial B^C_{1,+} \) and \( \lambda > 0 \) we have

$$\varphi(\lambda u) = \int_0^b G(\lambda u'(t)) \, dt - \int_0^b F(t,\lambda u(t)) \, dt$$

$$\leq C_{11} \left( 1 + \lambda^p \left\| u' \right\|^p_p \right) - \int_0^b F(t,\lambda u(t)) \, dt \text{ for some } C_{11} > 0 \text{ (see (2.3))} \quad (3.44)$$

$$= C_{11} \left( 1 + \lambda^p \left\| u' \right\|^p_p \right) - \int_0^b F(t,\lambda u^+(t)) \, dt - \int_0^b F(t,-\lambda u^-(t)) \, dt.$$ 

By virtue of \( H(f)(i),(ii) \), given any \( \xi > 0 \), we can find \( a_3 \in L^1(T) \) such that

$$F(t,x) \geq \xi x^p - a_3(t) \text{ for a.a. } t \in T, \text{ all } x \geq 0. \quad (3.45)$$

On the other hand, hypotheses \( H(f)(i),(iii) \) imply that there exist \( C_{12} > 0 \) and \( a_4 \in L^1(T) \) such that

$$F(t,x) \geq -C_{12} |x|^p - a_4(t) \text{ for a.a. } t \in T, \text{ all } x \leq 0. \quad (3.46)$$

Returning to (3.44) and using (3.45) and (3.46), we have

$$\varphi(\lambda u) \leq \lambda^p C_{11} \left\| u' \right\|^p_p - \lambda^p \xi \left\| u^+ \right\|^p_p + \lambda^p C_{12} \left\| u^- \right\|^p_p + C_{13}$$

for some \( C_{13} > 0 \)

$$\leq \lambda^p \left[ C_{11} \left\| u' \right\|^p_p + C_{12} \left\| u^- \right\|^p_p - \xi \left\| u^+ \right\|^p_p \right] + C_{13}. \quad (3.47)$$
Since $\xi > 0$ is arbitrary, we choose $\xi > 0$ big such that

$$C_{11} \| u' \|^p + C_{12} \| u^- \|^p < \xi \| u^+ \|^p.$$  

Then from (3.47) it follows that

$$\varphi (\lambda u) \to -\infty \text{ as } \lambda \to +\infty. \quad (3.48)$$

Hypotheses $\mathbf{H}(f) (ii), (iii)$ imply that there exist $\beta_2 \in (0, \beta_0), \ M_8 > 0$ and $\xi^* > 0$ such that

$$pF (t, x) - f (t, x) x \leq -\beta_2 x^\tau \text{ for a.a. } t \in T, \text{ all } x \geq M_8, \quad (3.49)$$

$$pF (t, x) - f (t, x) x \leq \xi^* \text{ for a.a. } t \in T, \text{ all } x \leq 0. \quad (3.50)$$

By (3.49) and (3.50) and hypothesis $\mathbf{H}(f) (i)$, for every $u \in W$, we have

$$\int_0^b [pF (t, u) - f (t, u) u] dt = \int_{\{u \leq 0\}} [pF (t, u) - f (t, u) u] dt + \int_{\{u \geq M_8\}} [pF (t, u) - f (t, u) u] dt + \int_{\{0 < u < M_8\}} [pF (t, u) - f (t, u) u] dt$$

$$\leq C_{14} - \beta_2 \int_{\{u \geq M_8\}} u^\tau dt. \quad (3.51)$$

Let $i : \widehat{C}^1 (T) \to W$ be the embedding map. Hence $i \in \mathcal{L} \left( \widehat{C}^1 (T), W \right)$. We see that

$$\psi = \varphi \circ i.$$

From the chain rule, we have

$$\psi' = i^* \varphi' (u) \text{ for all } u \in W. \quad (3.52)$$
Let \( \langle \cdot, \cdot \rangle_C \) denote the duality brackets for the pair \( (\widehat{C}^1 (T)^*, \widehat{C}^1 (T)) \). We have

\[
\frac{d}{d\lambda} \psi (\lambda u) = \langle \psi' (\lambda u), u \rangle_C \\
= \langle i^* \varphi' (\lambda u), u \rangle_C \quad \text{(see (3.52))} \\
= \langle \varphi' (\lambda u), u \rangle_C \\
= \frac{1}{\lambda} \left[ \int_0^b a (|\lambda u'|) (\lambda u')^2 \, dt - \int_0^b f (t, \lambda u) \lambda u \, dt \right] \\
\leq \frac{1}{\lambda} \left[ \int_0^b pG (\lambda u') \, dt - \int_0^b pF (t, \lambda u) \, dt + C_{14} \right] \\
\quad \text{(see \( \mathbf{H} (a) (iv) \) and (3.51))} \\
= \frac{1}{\lambda} [p \varphi (\lambda u) + C_{14}] \\
\tag{3.53}
\]

From (3.48) and (3.53), we see that for \( \lambda > 0 \) big (such that \( \varphi (\lambda u) < -\frac{C_{14}}{p} \)), we have

\[
\frac{d}{d\lambda} \psi (\lambda u) < 0. \\
\tag{3.54}
\]

From Proposition 3.3, we have

\[
\inf_{-\widehat{C}^1} \psi = \inf_{-\widehat{C}^1} \varphi > -C_{15} \text{ for some } C_{15} > 0.
\]

Let

\[
\alpha < \min \left\{ -C_{15}, -\frac{C_{14}}{p}, \inf_{\overline{B}^C_1} \psi \right\}
\]

where

\[
\overline{B}^C_1 = \left\{ u \in \widehat{C}^1 (T) : \|u\|_{\widehat{C}^1 (T)} \leq 1 \right\}.
\]

From (3.54) we see that we can find a unique \( \gamma (u) \geq 1 \) such that

\[
\psi (\lambda u) > \alpha \text{ if } \lambda < \gamma (u), \\
\psi (\lambda u) = \alpha \text{ if } \lambda = \gamma (u), \\
\psi (\lambda u) < \alpha \text{ if } \lambda > \gamma (u)
\]

and

\[
\psi^\alpha = \{ \lambda u : u \in \partial B^C_{1,+}, \lambda \geq \gamma (u) \}. \\
\tag{3.55}
\]
By virtue of the implicit function theorem, we have \( \gamma \in C(\partial B^C_{1,+}, [1, \infty)) \). Let
\[
V_+ = \{ \lambda u : u \in \partial B^C_{1,+}, \lambda \geq 1 \}.
\]
It is easily seen that \( \partial B^C_{1,+} \) is a retract of \( V_+ \) and \( V_+ \) is deformable onto \( \partial B^C_{1,+} \) in \( \hat{C}^1(T) \). Then invoking Dugundji [10] (Theorem 6.5, p.325), we infer that \( \partial B^C_{1,+} \) is a deformation retract of \( V_+ \). Therefore
\[
V_+ \text{ and } \partial B^C_{1,+} \text{ are homotopy equivalent.} \tag{3.56}
\]
We introduce the homotopy \( \hat{h}_+: [0,1] \times V_+ \rightarrow V_+ \) by
\[
\hat{h}_+(t, \lambda u) = \begin{cases} 
(1-t)\lambda u + t\gamma(u)u & \text{if } \lambda \in [1, \gamma(u)], \\
\lambda u & \text{if } \lambda \geq \gamma(u).
\end{cases}
\]
Then (cf. (3.55))
\[
\hat{h}_+(0,.) = Id, \quad \hat{h}_+(1, \lambda u) \in \psi^\alpha \text{ for all } \lambda u \in V_+,
\]
\[
\hat{h}_+(t, .) |_{\psi^\alpha} = Id |_{\psi^\alpha}.
\]
This means that \( \psi^\alpha \) is a strong deformation retract of \( V_+ \). Therefore
\[
V_+ \text{ and } \psi^\alpha \text{ are homotopy equivalent.} \tag{3.57}
\]
From (3.56) and (3.57) it follows that
\[
\psi^\alpha \text{ and } \partial B^C_{1,+} \text{ are homotopy equivalent},
\]
hence
\[
H_k\left(\hat{C}^1(T), \psi^\alpha\right) = H_k\left(\hat{C}^1(T), \partial B^C_{1,+}\right) \text{ for all } k \geq 0 \tag{3.58}
\]
(see Granas-Dugundji [14], p. 387). Recall that \( \partial B^C_{1,+} \) is contractible in itself. So,
\[
H_k\left(\hat{C}^1(T), \partial B^C_{1,+}\right) = 0 \text{ for all } k \geq 0 \tag{3.59}
\]
(see Granas-Dugundji [14], p. 389). From (3.58) and (3.59) it follows that
\[
H_k\left(\hat{C}^1(T), \psi^\alpha\right) = 0 \text{ for all } k \geq 0,
\]
therefore
\[
C_k(\varphi, \infty) = 0 \text{ for all } k \geq 0
\]
(see the first part of the proof).\[\square\]
Also, we compute the critical groups of \( \hat{\varphi}_+ \) at infinity.

**Proposition 3.6.** If hypotheses \( \mathbf{H}(a) \) and \( \mathbf{H}(f) \) hold, then \( C_k(\hat{\varphi}_+, \infty) = 0 \) for all \( k \geq 0 \).

**Proof.** By virtue of hypotheses \( \mathbf{H}(f) (i), (ii) \), given any \( \xi > 0 \), we can find \( a_5 \in L^1(T) \) such that

\[
F(t,x) \geq \xi x^p - a_5(t) \quad \text{for a.a. } t \in T, \text{ all } x \geq 0.
\]

We introduce the set

\[
E_+ = \{ u \in W : \|u\| = 1, u^+ \neq 0 \}.
\]

For \( u \in E_+ \) and \( \lambda > 0 \), we have

\[
\hat{\varphi}_+(\lambda u) = \int_0^b G(\lambda u'(t)) \, dt + \frac{\lambda^p}{p} \|u\|_p^p - \int_0^b \hat{F}_+(t,u) \, dt
\]

\[
\leq C_{16} \left( 1 + \lambda^p \|u'\|_p^p \right) + \frac{\lambda^p}{p} \|u\|_p^p - \xi \lambda^p \|u^+\|_p^p + C_{17}
\]

for some \( C_{16}, C_{17} > 0 \)

\[
= \lambda^p \left[ \|u'\|_p^p + \frac{1}{p} \|u\|_p^p - \xi \|u^+\|_p^p \right] + C_{18},
\]

with \( C_{18} = C_{16} + C_{17} > 0 \).

Since \( \xi > 0 \) is arbitrary, we choose \( \xi > 0 \) big such that

\[
\|u'\|_p^p + \frac{1}{p} \|u\|_p^p < \xi \|u^+\|_p^p.
\]

So, from (3.60) it is clear that

\[
\hat{\varphi}_+(\lambda u) \to -\infty \text{ as } \lambda \to +\infty.
\]

Similarly, as in the proof of Proposition 3.5 (see (3.51)), for all \( u \in W \) we have

\[
\int_0^b \left[ p\hat{F}_+(t,u) - f(t,u) \right] \, dt = \int_\{u>0\} \left[ pF(t,u) - f(t,u) \right] \, dt
\]

\[
= \int_{\{0<u<M_8\}} \left[ pF(t,u) - f(t,u) \right] \, dt
\]

\[
+ \int_{\{u\geq M_8\}} \left[ pF(t,u) - f(t,u) \right] \, dt, \text{ for some } M_8 > 0 \text{ big (see (3.49))}
\]

\[
\leq C_{19} - \beta_2 \int_{\{u\geq M_8\}} u^\tau \, dt, \text{ for some } C_{19} > 0
\]
Using (3.62), we have
\[
\frac{d}{d\lambda} \hat{\varphi}_+ (\lambda u) = \langle \hat{\varphi}'_+ (\lambda u), u \rangle \\
= \int_0^b \left( a (|\lambda u'|^p) \lambda (u')^2 dt + \frac{\lambda^{p-1}}{p} \|u\|^p_p - \int_0^b \hat{f}_+ (t, \lambda u) u dt \right) \\
= \frac{1}{\lambda} \left[ \int_0^b a (|\lambda u'|) (\lambda u')^2 dt + \lambda^p \|u\|^p_p - \int_0^b \hat{f}_+ (t, \lambda u) \lambda u dt \right] \\
\leq \frac{1}{\lambda} \left[ \int_0^b pG (\lambda u') dt + \lambda^p \|u\|^p_p - \int_0^b p\hat{F}_+ (t, \lambda u) \lambda u dt + C_{19} \right] \\
= \frac{1}{\lambda} \left[ p\hat{\varphi}_+ (\lambda u) + C_{19} \right].
\]
(see H(a) (iv) and (3.62))
So, for \( \lambda > 0 \) big (such that \( \hat{\varphi}_+ (\lambda u) < -\frac{C_{19}}{p} \)), we have
\[
\frac{d}{d\lambda} \hat{\varphi}_+ (\lambda u) < 0.
\]
Let \( d < -\frac{C_{19}}{p} \). We can find a unique \( \gamma_+ (u) > 0, \gamma_+ \in C (E_+) \) (by the implicit function theorem) such that
\[
\hat{\varphi}_+ (\gamma_+ (u) u) = d \text{ for all } u \in E_+.
\]
Let \( D_+ := \{u \in W : u^+ \neq 0\} \) and define
\[
\hat{\gamma}_+ (u) = \frac{1}{\|u\|} \gamma_+ \left( \frac{u}{\|u\|} \right) \text{ for all } u \in D_+.
\]
We have \( \hat{\gamma}_+ \in C (D_+) \), and from (3.63) it follows
\[
\hat{\varphi}_+ (\hat{\gamma}_+ (u) u) = d \text{ for all } u \in D_+.
\]
Moreover, if \( \hat{\varphi}_+ (u) = d \), then \( \hat{\gamma}_+ (u) = 1 \). We set
\[
\widetilde{\gamma}_+ (u) = \begin{cases} 
1 & \text{if } \hat{\varphi}_+ (u) \leq d \\
\hat{\gamma}_+ (u) & \text{if } \hat{\varphi}_+ (u) > d.
\end{cases}
\]
(3.65)
for all \( u \in D_+ \). We see that \( \widetilde{\gamma}_+ \in C (D_+) \).
We introduce the homotopy \( \tilde{h}_+ : [0, 1] \times D_+ \to D_+ \) by
\[
\tilde{h}_+ (t, u) = (1 - t) u + t \tilde{\gamma}_+ (u) u \quad \text{for all} \quad (t, u) \in [0, 1] \times D_+.
\]
We have
\[
\tilde{h}_+ (0, u) = u, \quad \tilde{h}_+ (1, u) \in \hat{\varphi}^d_+ \quad \text{and} \quad \tilde{h}_+ (t, .) |_{\hat{\varphi}^d_+} = Id |_{\hat{\varphi}^d_+}
\]
for all \((t, u) \in [0, 1] \times D_+\).

This shows that \( \hat{\varphi}^d_+ \) is a strong deformation retract of \( D_+ \). Hence
\[
D_+ \quad \text{and} \quad \hat{\varphi}^d_+ \quad \text{are homotopy equivalent},
\]
therefore
\[
H_k (W, D_+) = H_k \left( W, \hat{\varphi}^d_+ \right) \quad \text{for all} \quad k \geq 0. \tag{3.66}
\]
Without any loss of generality we assume that \( K_{\hat{\varphi}_+} \) is finite. (Otherwise we already have a sequence of distinct positive solutions of (1.1).) We choose
\[
d < \min \left\{ \inf \hat{\varphi}_+ \left( K_{\hat{\varphi}_+} \right), -\frac{C_{19}}{p} \right\}.
\]
Then we have
\[
H_k \left( W, \hat{\varphi}^d_+ \right) = C_k \left( \hat{\varphi}_+, \infty \right) \quad \text{for all} \quad k \geq 0. \tag{3.67}
\]
Consider the homotopy \( h^* : [0, 1] \times D_+ \to D_+ \) defined by
\[
h^* (t, u) = \frac{(1 - t) u + t \hat{u}_0}{\| (1 - t) u + t \hat{u}_0 \|}
\]
where \( \hat{u}_0 \in \text{int } \hat{C}_+ \). Then \( h^* (1, u) = \frac{\hat{u}_0}{\| \hat{u}_0 \|} \), which proves that \( D_+ \) is contractible in itself. Therefore
\[
H_k (W, D_+) = 0 \quad \text{for all} \quad k \geq 0 \tag{3.68}
\]
(see Granas-Dugundji [14], p.389). From (3.66), (3.67) and (3.68) we conclude that
\[
C_k \left( \hat{\varphi}_+, \infty \right) = 0 \quad \text{for all} \quad k \geq 0.
\]

Now we are ready to produce a third nontrivial solution and have the complete multiplicity theorem (three solution theorem) for problem (1.1).

**Theorem 3.7.** If hypotheses \( \mathbf{H} (a) \) and \( \mathbf{H} (f) \) hold, then problem (1.1) has at least three nontrivial solutions \( u_0 \in \text{int } \hat{C}_+, \, v_0 \in -\text{int } \hat{C}_+ \) and \( y_0 \in \hat{C}^1 (T) \).
Proof. From Proposition 3.4 we already have two constant sign solutions \(u_0 \in \text{int} \hat{C}_+\) and \(v_0 \in -\text{int} \hat{C}_+\). We may assume that \(K_{\hat{\varphi}_+} = \{0, u_0\}\), or otherwise we already have a third solution of (1.1) which in fact is positive.

Claim: \(C_k(\hat{\varphi}_+, u_0) = \delta_{k,1}\mathbb{Z}\) for all \(k \geq 0\). Here and in what follows \(\delta_{k,j} (k, j \in \mathbb{Z}_+)\) denotes the Kronecker delta. Let

\[ d < 0 = \hat{\varphi}_+(0) < \eta < \hat{n}_p^+ \leq \hat{\varphi}_+(u_0) \]

(see (3.32) and (3.34)). We consider the following triple of sets

\[ \hat{\varphi}_+^d \subseteq \hat{\varphi}_+^\eta \subseteq W. \]

For this triple, we consider the corresponding long exact sequence of homology groups

\[
\ldots \to H_k \left( W, \hat{\varphi}_+^d \right) \xrightarrow{i_*} H_k \left( W, \hat{\varphi}_+^\eta \right) \xrightarrow{\partial_*} H_{k-1} \left( \hat{\varphi}_+^\eta, \hat{\varphi}_+^d \right) \to \ldots, \tag{3.69}
\]

where \(i_*\) is the group homomorphism induced by the inclusion \( (W, \hat{\varphi}_+^d) \hookrightarrow (W, \hat{\varphi}_+^\eta) \) and \(\partial_*\) is the boundary homomorphism. We have

\[
H_k \left( W, \hat{\varphi}_+^d \right) = C_k \left( \hat{\varphi}_+, \infty \right) \text{ for all } k \geq 0 \text{ (see Proposition 3.6)}, \tag{3.70}
\]

\[
H_k \left( W, \hat{\varphi}_+^\eta \right) = C_k \left( \hat{\varphi}_+, u_0 \right) \text{ for all } k \geq 0 \text{ (recall that } K_{\hat{\varphi}_+} = \{0, u_0\}\text{)}, \tag{3.71}
\]

\[
H_{k-1} \left( \hat{\varphi}_+^\eta, \hat{\varphi}_+^d \right) = C_{k-1} \left( \hat{\varphi}_+, 0 \right) = \delta_{k-1,0}\mathbb{Z} = \delta_{k,1}\mathbb{Z} \text{ for all } k \geq 0 \tag{3.72}
\]

(see the proof of Proposition 3.4)).

From (3.70), (3.71), (3.72) and the exactness of (3.69), we see that in (3.69) only the tail of a long sequence (i.e., \(k = 1\)), is nontrivial. So, we focus on the tail and use the rank theorem to obtain

\[
\text{rank } C_1 \left( \hat{\varphi}_+, u_0 \right) = \text{rank } H_1 \left( W, \hat{\varphi}_+^\eta \right)
= \text{rank } \ker \partial_* + \text{rank } \text{Im } \partial_*
= \text{rank } \text{Im } i_* + \text{rank } \text{Im } \partial_* \text{ (by the exactness of (3.69) )}
\leq 0 + 1 \text{ (see (3.70), (3.72) )}. \tag{3.73}
\]

On the other hand, from the proof of Proposition 3.4, we know that \(u_0 \in \text{int} \hat{C}_+\) is a critical point of \(\hat{\varphi}_+\) of mountain pass type. Hence

\[
\text{rank } C_1 \left( \hat{\varphi}_+, u_0 \right) \geq 1. \tag{3.74}
\]
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From (3.73) and (3.74), it follows that

$$\text{rank } C_1(\hat{\varphi}_+, u_0) = 1,$$

hence

$$C_k(\hat{\varphi}_+, u_0) = \delta_{k,1}\mathbb{Z} \text{ for all } k \geq 0,$$

(3.75)
as claimed.

Since $\varphi|_{\hat{C}+} = \hat{\varphi}_+|_{\hat{C}+}$ and $u_0 \in \text{int } \hat{C}_+$, we have

$$C_k\left(\varphi|_{\hat{C}^1(T)}, u_0\right) = C_k\left(\hat{\varphi}_+|_{\hat{C}^1(T)}, u_0\right) \text{ for all } k \geq 0,$$

hence

$$C_k(\varphi, u_0) = C_k(\hat{\varphi}_+, u_0) \text{ for all } k \geq 0 \text{ (see Palais [15])},$$

therefore

$$C_k(\varphi, u_0) = \delta_{k,1}\mathbb{Z} \text{ for all } k \geq 0 \text{ (see (3.75))}.$$  

(3.76)
Recall that $v_0$ and 0 are local minimizers of $\varphi$ (see Proposition 3.4 and its proof). Therefore, we have

$$C_k(\varphi, v_0) = C_k(\varphi, 0) = \delta_{k,0}\mathbb{Z} \text{ for all } k \geq 0.$$  

(3.77)
Finally, from Proposition 3.5, we have

$$C_k(\varphi, \infty) = 0 \text{ for all } k \geq 0.$$  

(3.78)
Suppose that $K_\varphi = \{0, v_0, u_0\}$. From (3.76), (3.77), (3.78) and the Morse relation with $t = -1$ (see (2.1)), we infer that

$$2(-1)^0 + (-1)^1 = 0,$$

which is a contradiction.

So, there exists $y_0 \in K_\varphi, y_0 \notin \{0, v_0, u_0\}$.

Therefore $y_0$ is the third nontrivial solution of (1.1) and $y_0 \in \hat{C}^1(T)$.

\[\Box\]

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References


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