ON THE CODIMENSION $n$ MINIMAL SURFACES CARRYING A
COVARIANT DECOMPOSABLE TANGENTIAL VECTOR FIELD
IN A RIEMANNIAN OR PSEUDO-RIEMANNIAN SPACE FORM

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INTRODUCTION

Let $x : M \to \tilde{M}(k)$ be the immersion of a codimension $n$ surface in a Riemannian space form $\tilde{M}(k)$. If $T(M)$ and $T^4(M)$ are the vector tangent bundle and the vector normal bundle on $M$, then the tangent bundle of $M$ restricted to $M$, is the direct sum $T(M)/M = T(M) \oplus T^4(M)$.

Let then $X \in T(M)$ and $\nabla$ be a tangent vector on $M$, and the covariant differential operator on $M$, respectively. We say that $X$ is covariant decomposable if it satisfies $X = u \otimes T + v \otimes N$, where $T \in T^4(M)$, $N \in T_p^1(M)$ $(p \in M)$, and $u, v \in \Lambda^1(M)$ are two unit 1-forms.

We agree to call $T$ and $N$ the tangential and the normal vector components of $\nabla X$, and $u$ and $v$ the tangential and the normal Pfaffian components of $\nabla X$.

If $M$ is minimal and $r$ is the scalar curvature of $M$, then one finds $r = -2 |N|^2/|X|^2 + 2k$. In particular if $M$ is closed, it follows according to Simon's theorem [1], $r = 2k - \frac{2n}{2n-1}$.

Next one considers the case where $M$ is a spatial minimal surface of a hyperbolic space form $M(k)$. If the normal vector component of $\nabla X$ is a null real vector field, it follows that the scalar curvature of $M$ is $2k$.

Finally, let $\tilde{M}(\tilde{\phi}, \tilde{n}, \tilde{g}, \tilde{\xi}, k)$ be a 5-dimensional Sasakian space-form, with structure tensor fields $\tilde{\phi}$, $\tilde{n}$, $\tilde{g}$, $\tilde{\xi}$, and let $M$ be an anti-invariant [2] minimal surface of $\tilde{M}$, (the tangent space of $M$ is mapped by $\tilde{\phi}$ into the normal space). We say that a tangential vector field $X$ is covariant $\tilde{\phi}$-decomposable if
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\[ \nabla X = u \otimes T + v \otimes \phi X \] (one has the decomposition: \( T_p^r(M) = \phi T_p(M) \oplus N_p(M) \)). In this case, it is proved that the scalar curvature of \( M \) is \((k-1)/2\).

SECTION I

Let \( x : (M, g) \to (\tilde{M}, \tilde{g}) \) be the immersion of the \( C^\infty \)-surface \( M \) in a Riemannian \((n+2)-C^\infty\)-manifold \( \tilde{M} \). Let \( F(M) \) and \( F(\tilde{M}) \) be the bundles of orthonormal frames of \( M \) and \( \tilde{M} \) respectively; if \( p \in M \), let \( B \) be the set of elements \( B = (p, e_A^i : A, B \in \{1, \ldots, n+2\}) \) such that \( (p, e_i^j : i, j \in \{1, 2\}) \in F(M) \) and \((x(p), e_A^i) \in F(\tilde{M})\) whose orientation is coherent with the one of \( \tilde{M} \). Let \( \omega^i \) be the dual forms of \( e_i \), induced by \( x \), and \( \omega^A_B = \gamma^A_B i^i \) \((\gamma^A_B \in C^\infty(M))\) the connection forms induced by \( x \). Then the line element \( dp \) (\( dp \) is a canonical differential vectorial 1-form on \( F(M) \)) is expressed by:

\[ dp = \omega^i \otimes e_i^i. \]  

(1.1)

If \( V \) is the covariant differentiation defined by \( g \), the structure equations (E. Cartan) of \( M \) are given by:

\[ \nabla e_i^A = \omega_i^A \otimes e_A^i, \]  

(1.2)

\[ d\omega^i = \omega^j \wedge \omega^i_j, \]  

(1.3)

\[ d\omega^A_B = \omega^C_B \wedge \omega^A_C + \Omega^A_B, \]  

(1.4)

where \( \Omega^A_B \) are the curvature 2-form induced by \( x \).

SECTION II

If \( T_p(M) \) is the tangent plane at \( p \in M \), let

\[ X = \sum_i x_i e_i^i \in T_p(M); \quad x_i \in C^\infty(M) \]  

(2.1)

be a tangent vector field on \( M \). If the covariant differential of \( X \) may be expressed as:
\[ \nabla X = u \otimes T + v \otimes N \] (2.2)

where \( u, v \in \Lambda^1(M) \) are two unitary 1-forms, while \( T \in T_p(M) \) and \( N \in T^*_p(M) \) are a tangent vector field and a normal vector field, respectively, we say that \( X \) is covariant decomposable.

Set

\[ T = \sum_i t_i e_i; \quad t_i \in C^\infty(M), \] (2.3)

\[ N = \sum_r \mu_r e_r; \quad r \in \{3, \ldots, n+2\}; \quad \mu_r \in C^\infty(M), \] (2.4)

\[ u = u_i^j \omega^i_j, \quad v = v_i^j \omega^i_j; \quad u_i^j, v_i^j \in C^\infty(M). \] (2.5)

We agree to call \( T \) and \( N \) the tangential and the normal vector components of \( \nabla X \), and \( u \) and \( v \) the tangential and the normal Pfaffian components of \( \nabla X \). One has:

\[ \sum_i (u_i^j)^2 = 1, \quad \sum_i (v_i^j)^2 = 1 \] (2.6)

and making use of (1.2) one gets:

\[ dx_i + x_j^i \omega^i_j = t_i u, \] (2.7)

\[ \sum_j x_i^j \gamma^r_{ij} = v_i^j u_r. \] (2.8)

By Cartan's lemma one has:

\[ \gamma^r_{ij} = \gamma^r_{ji}. \] (2.9)

As known from \([3]\),

\[ h = \sum \gamma^r_{ij} \omega^i_j \otimes \omega^j_i \otimes e_r \in (T^* \otimes T^*)^l(M) \] (2.10)

and

\[ H = \frac{1}{2} \sum_r (\gamma^r_{11} + \gamma^r_{22}) e_r \in T^*_p(M) \] (2.11)

represent the second fundamental vectorial form \([3]\) and the mean curvature vector associated with \( x \), respectively (h is an
M-morphism of $T_0^2(M)$ in $T(M)$ and is independent of the normal bundle $T(M) - UT_p(M)$. In the following, we shall suppose that $M$ is minimal, that is

$$H = 0 \iff \mu_{11}^r + \mu_{22}^r = 0, \text{ for all } r \in \{3, \ldots, n+2\}, \quad (2.12)$$

and that $\tilde{M}$ is a space form of curvature $k$. Then by (2.9), (2.12) and (2.8) a straightforward calculation gives:

$$|X|^2 \gamma_{ij}^r = \mu_r(v_i x_j - v_j x_i) \quad (2.13)$$

$$|X|^2 \gamma_{ij}^r = \mu_r(v_i x_j + v_j x_i).$$

Let $\langle h \rangle$ denote the length of the second fundamental form $h$, that is $[4]$

$$\langle h \rangle^2 = \sum_{r,i,j} (\gamma_{ij}^r)^2. \quad (2.14)$$

Making use of (2.13) and taking account of (2.6) one derives

$$\langle h \rangle^2 = 2 \frac{|N|^2}{|X|^2}. \quad (2.15)$$

On the other hand since $\tilde{M}$ is a space form of curvature $k$, the scalar curvature $r$ of $M$ satisfies the following general relation $[4]$ (if $m$ is the dimension of $M$):

$$r = m^2 |H|^2 - \langle h \rangle^2 + m(m-1)k. \quad (2.16)$$

Therefore, in the case under discussion we have:

$$r = -2 \frac{|N|^2}{|X|^2} + 2k. \quad (2.17)$$

So, it follows that the necessary and sufficient condition for $M$ to be of constant scalar curvature, is that $|N|/|X| = \text{const.}$

Since $M$ is not totally geodesic it is worth remarking that if $M$ is a unit $(n+2)$-sphere, and $M$ is closed (there exists no closed minimal submanifold in a space form of nonpositive curvature)
then according to Simon's theorem [1] we have $|N|^2/|x|^2 = n/2n-1$ 
$\Rightarrow r = 2k - 2n/2n-1$. Denote

$$dx_1 = x_{1,j}\omega^j,$$  \hspace{1cm} (2.18)

where $x_{1,j}$ means the Pfaffian derivative, and suppose that $X$ 
is a gradient vector field. This condition is expressed in 
intrinsic manner by:

$$\langle V_{Z',Z}, X, X \rangle = 0,$$  \hspace{1cm} (2.19)

for all $Z, Z' \in T_p(M)$. Referring to (2.1) one finds from (2.19) the following relation

$$x_{1,2} - x_{2,1} + x_{2,1}^{12} - x_{1}^{11} = 0.$$  \hspace{1cm} (2.20)

So, taking account of (2.7), it readily follows with the help of 
(2.20)

$$X = \lambda j^{-1}(u), \lambda \in C^\infty(M)$$  \hspace{1cm} (2.21)

where $j$ is the canonical isomorphism defined by $g$.

Accordingly we have the following result.

**Theorem.** Let $x : M \to \tilde{M}$ be the immersion of a minimal surface 
in an $(n+2)$-dimensional space form $\tilde{M}$ of curvature $k$. If $M$ 
carries a tangential covariant decomposable vector field $X$, and $N$ 
and $u$ are the normal vectorial component of $\nabla X$ and the 
tangential Pfaffian component of $\nabla X$, respectively, then:

(i) the scalar curvature of $M$ is expressed by 
$2(2k - |N|^2/|x|^2)$;

(ii) if $X$ is a gradient vector field, then it is colinear 
to the dual vector field of $u$.

**SECTION III**

Consider now the immersion $x : (M,g) \to (\tilde{M},\tilde{g})$ where $M$ is 
a hyperbolic space form (with normal signature) of curvature $k$, 
and $\tilde{M}$ a spatial surface. We suppose as in Section II, that $\tilde{M}$ 
is of dimension $n+2$, and that $M$ is defined by

$$\omega^i = 0.$$  \hspace{1cm} (3.1)
With respect to an orthonormal basis, we denote by $r, s \in \{3, \ldots, n+2\}$ the normal indices and by $\alpha, \beta \in \{1, 2, \ldots, n+1\}$ the spatial indices ($\langle e_{n+2}, e_{n+2} \rangle = 1$, $\langle e_{\alpha}, e_{\alpha} \rangle = -1$). Then according to [5], the line element $dp$ of $M$ is
\begin{equation}
   dp = -\omega^i \otimes e_i, \tag{3.2}
\end{equation}
and the structure equations are
\begin{equation}
   \nabla e_{\alpha} = \alpha^A_{\alpha} \otimes e_A; \quad A \in \{1, \ldots, 2n+2\}, \tag{3.3}
\end{equation}
\begin{equation}
   \nabla e_{n+2} = -\omega^{n+2}_{n+2} \otimes e_{\alpha}, \tag{3.4}
\end{equation}
\begin{equation}
   d\omega^1 = \omega^j \wedge \omega^1_j, \tag{3.5}
\end{equation}
\begin{equation}
   d\omega^\alpha = \omega^\alpha_{\beta} + \varepsilon_{\alpha \beta} \otimes \omega^{\alpha}_{\beta}, \quad \varepsilon_{\alpha} = 1, \quad \varepsilon_{n+2} = -1,
\end{equation}
\begin{equation}
   d\omega^{n+2} = \omega^{n+2}_{\alpha} + \omega^{n+2}_{\beta} \wedge \omega^{n+2}_{\beta}. \tag{3.6}
\end{equation}

If $X \in T_p(M)$ is a tangential vector field we suppose that $X$ satisfies equation (2.2) (i.e. $X$ is covariant decomposable) and that $M$ is minimal. Since the mean curvature vector $H$, associated with $X$, is expressed by [5]
\begin{equation}
   H = \Sigma_p (\gamma^r_{11} + \gamma^r_{22}) e_r, \tag{3.7}
\end{equation}
condition $H = 0$, leads as in [2] to:
\begin{equation}
   \langle h \rangle^2 = 2 \frac{|N|^2}{|X|^2}, \tag{3.8}
\end{equation}
where
\begin{equation}
   N = \Sigma_p \mu_r e_r. \tag{3.9}
\end{equation}
Taking account of the signature of $\tilde{g}$, one has
\begin{equation}
   |N|^2 = -\mu_3^2 \cdots -\mu_{n+1}^2 + \mu_{n+2}^2, \tag{3.10}
\end{equation}
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and if $|N|^2 = 0$, then $N$ is a null real vector field. On the other hand, the general formula 2.16, still holds good for the immersion $x: M \to \tilde{M}$. Therefore if $N$ is the null real vector field, one may have $\langle h \rangle^2 = 0$, without $M$ to be totally geodesic [4]. Consequently we derive:

$$r = 2k,$$  \hspace{1cm} (3.9')

and so $M$ is of constant scalar curvature.

Thus we have the following

**Theorem.** Let $x: M \to \tilde{M}$ be the immersion of a minimal spatial surface $M$ in a hyperbolic space form $\tilde{M}$ of curvature $k$. If $M$ carries a tangential covariant decomposable vector field $x$, and the normal vectorial component of $\nabla x$ is a null real vector field, then the scalar curvature of $M$ is $2k$.

**SECTION IV**

We shall finally consider the immersion $x: M \to \tilde{M}$, where $\tilde{M}(\hat{\phi}, \hat{h}, \tilde{\xi})$ is a Sasakian space form of curvature $k$ (or of constant $\phi$-sectional curvature $k$) and $\tilde{M}$ is an anti-invariant [2] surface normal to the canonical vector field $\xi$. We recall that $\tilde{M}(\hat{\phi}, \hat{h}, \tilde{\xi})$ is of dimension $2m + 1$, then the structure tensors satisfy:

$$\hat{\phi}^2 Z = -Z + \hat{h}(Z)\xi, \quad \hat{\phi}^* = 0, \quad \hat{h}(\hat{\phi} Z) = 0, \quad \hat{h}(\xi) = 1,$$

$$d(\hat{h}(Z, Z')) = 2(\hat{\phi} Z, Z'), \quad \tilde{\nabla}_Z \xi = \hat{\phi} Z, \tilde{\nabla}_\xi = \hat{\phi} d\tilde{p},$$ \hspace{1cm} (4.1)

where $\tilde{\nabla}$ is the operator of covariant differentiation on $\tilde{M}$, $(\tilde{p} \in \tilde{M})$, and $Z, Z'$ are any vector fields on $\tilde{M}$.

One may take on $\tilde{M}$ a field of orthonormal frames $F(\tilde{M})$, such that if $F = \{e_A; A = 0, 1, \ldots, 2m\} \in F(\tilde{M})$, is an element of $F(\tilde{M})$, then $e_A = e_0 e_a e_{a^*}$; $a = 1, \ldots, m$; $a^* = a + m$, such that $e_0 = \xi$, $e_{a^*} = \phi e_a$ (a $\phi$-vector basis). If $\tilde{h} = \tilde{\omega}_a \tilde{\omega}^{a^*}$ is the dual basis, then the connections form $\omega^A_B = \gamma_{BC}^A \omega^C$ satisfy:

$$\omega^a_B = \tilde{\omega}^{a^*}_B, \quad \omega^{a^*}_a = \tilde{\omega}^b_{a^*}, \quad \omega^a_0 = \tilde{\omega}^a_0 = -\tilde{\omega}^{a^*}. \hspace{1cm} (4.2)$$
In the following, we assume that $\tilde{M}(k)$ is a 5-dimensional Sasakian space form and $M$ is a $C$-totally real minimal surface of $\tilde{M}(k)$ (in the sense of S. Yamaguchi, M. Kon, and Y. Miyahara [6]).

At each point $p \in M$ one has the decomposition $T^p(M) = \Phi T^p(M) \oplus N_p(M)$ where $N_p(M)$ is the orthogonal complement of $\Phi T^p(M)$ in the normal space $T^p(M)$.

Let then $X = \sum_{i} t_i e_i$ be a tangent vector field on $M$. We say that $X$ is covariant $\Phi$-decomposable if it satisfies:

$$\nabla X = u \otimes T + v \otimes \Phi X.$$  \hspace{1cm} (4.3)

Since $\Phi X = \sum_{i} t_i e_i$, clearly one has:

$$|X|^2 = |\Phi X|^2.$$  \hspace{1cm} (4.4)

On the other hand, $M$ being minimal, the scalar curvature $r$ of $M$ is according to [2], expressed by:

$$r = \frac{k+3}{2} - \sum_{i,j} (\nabla^r)_{ij}^2; \quad r = 3, 4.$$ \hspace{1cm} (4.5)

With the aid of $\nabla e_A = \omega^B_A \otimes e_B$, and taking account of (4.4), we get after some calculation from (4.3):

$$\sum_{i,j} (\nabla^r)_{ij}^2 = 2 \frac{|\Phi X|^2}{|X|^2} = 2.$$

Hence $r = k-1/2$, and we may formulate the result:

**Theorem.** Let $x : M \rightarrow \tilde{M}(k)$ be the immersion of a $C$-totally minimal surface $M$ in a 5-dimensional Sasakian space form $\tilde{M}(k)$. If $M$ carries a covariant $\Phi$-decomposable tangential vector field, then the scalar curvature of $M$ is $(k-1)/2$.

**REFERENCES**


