Ray Optics On Surfaces

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Abstract. Variational techniques are used to find path of light rays on surfaces. Using Fermat's principle of least time the problem is treated as a constraint optimization to obtain a system of partial differential equations.

Introduction. The time required for a beam of light to traverse a path is called the optical length of the curve. Fermat's principle states that in an optical medium, the path of light from a point $A$ to a point $B$ has the least optical length of all paths joining $A$ and $B$. Lyusternik [2, Chapter 6] gives an elementary treatment of Fermat's principle and its consequences. Goldstein [1] has a historical treatment of Fermat's principle. Weinstock [4, Chapter 5] discusses Fermat's principle and applications in geometric optics. In this article we use variational techniques to find path of light rays constraint on a given surface. By using Fermat's principle of least time we obtain a system of partial differential equations.

Discussion. First, as a review, we give differential equations for path of light rays in nonhomogeneous media. This is done in Whitham [5, pgs. 247-249]. For simplicity the proof is given in three dimensions.

Assume $c = c(x, y, z)$ is speed of light in the medium. Let $\sigma$ denote the total time travelled. Then

$$\sigma = \int \frac{ds}{dt} = \int \frac{\sqrt{x'^2 + y'^2 + z'^2}}{c(x, y, z)} \, dt,$$

where dot denotes differentiation with respect to the arc-length $s$. Let
\[ F(x, y, z, \dot{x}, \dot{y}, \dot{z}) = \frac{\sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}}{c(x, y, z)} \]

Using Euler-Lagrange equations in parametric form, we obtain

(1) \[ F_x - \frac{d}{dt} F_{\dot{x}} = 0 \]

(2) \[ F_y - \frac{d}{dt} F_{\dot{y}} = 0 \]

(3) \[ F_z - \frac{d}{dt} F_{\dot{z}} = 0 \]

Then from (1) and expression for \( F(x, y, z, \dot{x}, \dot{y}, \dot{z}) \) we have

\[- \frac{\partial c}{\partial x} \cdot \frac{x}{\sqrt{x^2 + y^2 + z^2}} - \frac{d}{dt} \left( \frac{x}{\sqrt{x^2 + y^2 + z^2}} \cdot \frac{1}{c} \right) = 0 \]

Now we use the chain rule

\[- \frac{\partial c}{\partial x} \cdot \frac{x}{\sqrt{x^2 + y^2 + z^2}} - \frac{d}{ds} \left( \frac{x}{\sqrt{x^2 + y^2 + z^2}} \cdot \frac{1}{c} \right) \sqrt{x^2 + y^2 + z^2} = 0. \]

Note that \( \frac{\dot{x}}{\sqrt{x^2 + y^2 + z^2}} = \frac{dx}{dT} = \frac{dx}{ds} \). Then we obtain

(4) \[ \frac{\partial c}{\partial x} \cdot \frac{1}{c^2} + \frac{d}{ds} \left( \frac{1}{c} \frac{dx}{ds} \right) = 0. \]

Similarly for \( y \) and \( z \) we obtain

(5) \[ \frac{\partial c}{\partial y} \cdot \frac{1}{c^2} + \frac{d}{ds} \left( \frac{1}{c} \frac{dy}{ds} \right) = 0, \]

(6) \[ \frac{\partial c}{\partial z} \cdot \frac{1}{c^2} + \frac{d}{ds} \left( \frac{1}{c} \frac{dz}{ds} \right) = 0. \]

The general form in \( n \) dimension with \( c = c(x_1, \ldots, x_n) \) is given by

(7) \[ \frac{\partial c}{\partial x_i} \cdot \frac{1}{c^2} + \frac{d}{ds} \left( \frac{1}{c} \frac{dx_i}{ds} \right) = 0. \]
Note that if $c$ is constant rays are straight lines as expected.

Now we use ray optics on surfaces as a constraint optimization by minimizing $\sigma$ given by

$$\sigma = \int \frac{ds}{c(x, y, z)} = \int \frac{\sqrt{x'^2 + y'^2 + z'^2}}{c(x, y, z)} \, dt$$

subject to a surface $G(x, y, z) = 0$. We use Lagrange multiplier $\lambda(t)$. Let

$$F(x, y, z, x', y', z', \lambda) = \frac{\sqrt{x'^2 + y'^2 + z'^2}}{c(x, y, z)} + \lambda(t)G(x, y, z).$$

See page 273 of Simmons [3] for this kind of Constraint optimization. We use Euler-Lagrange equation in parametric form. After simplification we obtain the following system of partial differential equations

(8) $$\frac{1}{c^2} \frac{\partial c}{\partial x} + \frac{d}{ds} \left( \frac{1}{c} \frac{dx}{ds} \right) = \lambda(t) \frac{G_x}{\sqrt{x'^2 + y'^2 + z'^2}},$$

(9) $$\frac{1}{c^2} \frac{\partial c}{\partial y} + \frac{d}{ds} \left( \frac{1}{c} \frac{dy}{ds} \right) = \lambda(t) \frac{G_y}{\sqrt{x'^2 + y'^2 + z'^2}},$$

(10) $$\frac{1}{c^2} \frac{\partial c}{\partial z} + \frac{d}{ds} \left( \frac{1}{c} \frac{dz}{ds} \right) = \lambda(t) \frac{G_z}{\sqrt{x'^2 + y'^2 + z'^2}}.$$

If we assume $c$ is constant we obtain the following equations for geodesics given in Simmons [3, p 374] as a special case

$$\frac{d^2 x}{ds^2} = \frac{d^2 y}{ds^2} = \frac{d^2 z}{ds^2} = G_x = G_y = G_z.$$

Also, if we let $f = \sqrt{x'^2 + y'^2 + z'^2}$ we can rewrite the above equations as

$$\frac{d}{dt} \left( \frac{x'}{f} \right) = \frac{d}{dt} \left( \frac{y'}{f} \right) = \frac{d}{dt} \left( \frac{z'}{f} \right).$$

REFERENCES


