ON THE UNIQUE SOLVABILITY OF SEMILINEAR PROBLEMS WITH STRONGLY MONOTONE NONLINEARITY

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Abstract: It is presented a method to solve semilinear equations in real Hilbert spaces. Some applications to differential equations are given.

1. INTRODUCTION

In [1] it is studied semilinear equations of the form

\[ Au = F(u) \]  

in a real Hilbert space $H$, where $A : D(A) \subset H \to H$ is a self-adjoint linear operator with the resolvent set $\rho(A)$ and $F : H \to H$ is a Gateaux differentiable gradient operator. In particular it is known that equation (1) possesses multiple solutions if the nonlinearity $F$ interacts suitably with the spectrum of $A$. In [2] it is presented the following existence and uniqueness theorem, as a corollary to some general considerations on saddle points:

**Theorem 1 (Amann).** Suppose that there exists real numbers $\nu < \mu$ such that $[\nu, \mu] \subset \rho(A)$ and

\[ \nu \leq \frac{\langle F(u) - F(v), u - v \rangle}{|u - v|^2} \leq \mu, \quad \forall \ u, v \in H, u \neq v. \]  

Then the equation $Au = F(u)$ possesses exactly one solution.

In this paper we consider the equation (1) of the form

\[ Au + F(u) = 0. \]  

We establish an existence and uniqueness result for (3) asking a condition of type (2) for $F$ and maximal monotony for $A$, but giving up from self-adjointness of $A$ and Gateaux differentiability of $F$. The condition of maximal monotony for $A$ is not very restrictive because the most known differential equations have this property.
2. THE MAIN RESULT

We give the following

Theorem 2. Assume that $A : D(A) \subset H \to H$ is maximal monotone and there exist $m, M > 0$ such that

(i) $\langle F(u) - F(v), u - v \rangle \geq m \cdot |u - v|^2$, $\forall u, v \in H$;
(ii) $|F(u) - F(v)| \leq M \cdot |u - v|$, $\forall u, v \in H$.

Then the equation (3) has an unique solution.

Proof: We shall use the following known result:

Lemma. Suppose that $F : H \to H$ satisfy (i) and (ii). Then there exists $\lambda > 0$ such that $S_{\lambda} : H \to H$, $S_{\lambda}(u) := u - \lambda F(u)$ is a contraction.

Indeed,

$$|S_{\lambda}(u) - S_{\lambda}(v)|^2 = |u - v|^2 - 2\lambda \langle F(u) - F(v), u - v \rangle + \lambda^2 |F(u) - F(v)|^2 \leq$$

$$\leq (1 - 2\lambda m + \lambda^2 M) |u - v|^2,$$

thus

$$(4) \quad |S_{\lambda}(u) - S_{\lambda}(v)| \leq c \cdot |u - v|,$$

with $c := \sqrt{1 - 2\lambda m + \lambda^2 M} < 1$, if $\lambda \in (0, \frac{2m}{M})$.

Now equation (3) can be written as

$$(5) \quad (I + \lambda A)u - (u - \lambda F(u)) = 0,$$

or

$$(6) \quad (I + \lambda A)u = S_{\lambda}(u),$$

where $\lambda > 0$ is taken from the lemma. Using the fact that $(I + \lambda A)$ is inversable and $|(I + \lambda A)^{-1}| \leq 1$ for each $\lambda > 0$ (because $A$ is maximal monotone, e.g.[3],p.101) the equation (6) is equivalent with

$$(7) \quad u = (I + \lambda A)^{-1}S_{\lambda}(u).$$

We have

$$|(I + \lambda A)^{-1}S_{\lambda}(u) - (I + \lambda A)^{-1}S_{\lambda}(v)| = |(I + \lambda A)^{-1}(S_{\lambda}(u) - S_{\lambda}(v))| \leq$$

$$\leq |(I + \lambda A)^{-1}| \cdot |S_{\lambda}(u) - S_{\lambda}(v)| \leq c \cdot |u - v|, u, v \in H.$$

Therefore, $u \mapsto (I + \lambda A)^{-1}S_{\lambda}(u)$ is a contraction having an unique fixed point, thus (7) and consequently (3) has an unique solution.$\square$
A similar result can be proved in the next case:

**Theorem 3.** Suppose that $F$ satisfy (i)+(ii) and $A : D(A) \subset H \to H$ is bounded, compact and monotone. Then the equation (3) has an unique solution.

**Proof:** Equation (3) can be equivalently written as

(8) \[(\lambda I + A)u = T_\lambda(u),\]

where $T_\lambda(u) := \lambda u - F(u)$, $\lambda > 0$. We have

\[
|T_\lambda(u) - T_\lambda(v)|^2 = \lambda^2 |u - v|^2 - 2\lambda < F(u) - F(v), u - v > + |F(u) - F(v)|^2 \leq \\
\leq (\lambda^2 - 2\lambda m + M^2) |u - v|^2,
\]

therefore

(9) \[|T_\lambda(u) - T_\lambda(v)| \leq \sqrt{\lambda^2 - 2\lambda m + M^2} \cdot |u - v|.
\]

Let us choose $\lambda > \max\{\|A\|, \frac{M^2}{2m}\}$. In particular, $\lambda > \|A\|$ imply that $\lambda I + A$ is inversible because $\sigma(A) \subset [-\|A\|, \|A\|]$. Moreover,

(10) \[|\lambda I + A|u|^2 = \lambda^2 |u|^2 + 2\lambda (Au, u) + |Au|^2 \geq \lambda^2 |u|^2,
\]

(because $A$ is monotone), or

\[|\lambda I + A|u| \geq \lambda |u|,
\]

hence $|(\lambda I + A)^{-1}| \leq \frac{1}{\lambda}$. Equation (8) is equivalent with

(11) \[u = (\lambda I + A)^{-1}T_\lambda(u).
\]

We have

\[
|(\lambda I + A)^{-1}T_\lambda(u) - (\lambda I + A)^{-1}T_\lambda(v)| = |(\lambda I + A)^{-1}(T_\lambda(u) - T_\lambda(v))| \leq \\
\leq |(\lambda I + A)^{-1}| \cdot |T_\lambda(u) - T_\lambda(v)| \leq \frac{1}{\lambda} \sqrt{\lambda^2 - 2\lambda m + M^2} \cdot |u - v|.
\]

Because $\lambda > \frac{M^2}{2m}$, it results that $\gamma := \frac{1}{\lambda} \sqrt{\lambda^2 - 2\lambda m + M^2} < 1$, therefore $u \mapsto (\lambda I + A)^{-1}T_\lambda(u)$ is a contraction. Now equation (11) and consequently (3) has an unique solution.

**Remark.** Compactness and boundedness of $A$ was used to choose a number $\lambda > 0$ such that $\lambda I + A$ is inversible. This is possible in weaker hypotesis. Indeed, the condition "A compact and bounded" can be replaced with "spectrum of $A$ is bounded". We can state the more general result:

**Theorem 3.** Let $F : H \to H$ satisfy (i)+(ii) and $A : D(A) \subset H \to H$ be monotone and the spectrum $\sigma(A)$ is bounded from below. Then equation (3) has an unique solution.
Indeed, it can be repeated the proof from theorem 2 taking \( \lambda > \frac{M^2}{2m} \) such that

\(-\lambda \in \rho(A)\).

3. APPLICATIONS

(A1). SEMILINEAR ELLIPTIC BOUNDARY PROBLEMS

Let \( \Omega \subset \mathbb{R}^N \) be a bounded domain and \( a_{ij} \in C^1(\overline{\Omega}), \ 1 \leq i, j \leq N \) having the ellipticity property

\[
\sum_{i,j=1}^{N} a_{ij}(x) \xi_i \xi_j \geq \alpha |\xi|^2, \ \forall \ \xi \in \mathbb{R}^N
\]

for some \( \alpha > 0 \). Let us consider the following elliptic problem

\[
\begin{aligned}
- \sum_{i,j=1}^{N} \frac{\partial}{\partial x_j} \left( a_{ij}(x) \frac{\partial u}{\partial x_i} \right) + g(x, u) &= f(x) \quad \text{in} \ \Omega \\
\frac{\partial u}{\partial n} &= 0 \quad \text{on} \ \partial\Omega
\end{aligned}
\]

where the nonlinearity is given by the real valued function \( f \in L^2(\Omega) \).

The particular case when \( g(x, u) = a_0(x)u \), with \( a_0 \in C(\overline{\Omega}), \ a_0 > p > 0 \) is studied in [3], p.177 using Lax-Milgram theorem and in [1], p.165 using the above theorem 1. Now we suppose that \( g(x, u) \) has partial derivative in \( u \) of the first order and

\[
m \leq \frac{\partial g}{\partial u} \leq M \quad \text{in} \ \Omega, \ (m, M > 0).
\]

Under these hypothesis, problem (12) has an unique solution in weak sense, for every \( f \in L^2(\Omega) \). Indeed, we can apply theorem 2 for the following functional background:

\[
H = L^2(\Omega), \quad Au := - \sum_{i,j=1}^{N} \frac{\partial}{\partial x_j} \left( a_{ij}(x) \frac{\partial u}{\partial x_i} \right), \quad D(A) := H^2(\Omega) \cap H_0^1(\Omega),
\]

\( F(u) := g(\cdot, u) - f \). \( A \) is monotone:

\[
(Au, u) = \int_{\Omega} \sum_{i,j=1}^{N} a_{ij} \frac{\partial u}{\partial x_j} \frac{\partial u}{\partial x_i} \geq 0
\]

and \( I + A \) is surjective ([3], p.177), thus \( A \) is maximal monotone. The conditions (i) and (ii) follows from (13).
(A2). In [5] is studied the perturbed Laplace problem

\begin{equation}
\begin{aligned}
-\Delta u + Pu &= f & \text{in } \Omega \\
u &= 0 & \text{on } \partial \Omega 
\end{aligned}
\end{equation}

using the variational theorem of Langenbach. We can apply theorem, asking that $P : L^2(\Omega) \to L^2(\Omega)$ satisfy (i) and (ii). In particular, if $P$ is Gateaux differentiable with

$$m \cdot |h|^2 \leq \langle (DP)(u)h, h \rangle \leq M \cdot |h|^2, \quad (m, M > 0)$$

then (14) has an unique solution, because $Au := -\Delta u$, $D(A) := H^2(\Omega) \cap H^1_0(\Omega)$ is maximal monotone.

(A3). PERIODIC SOLUTIONS OF SEMILINEAR WAVE EQUATION

Suppose that $A : D(A) \subset H \to H$ is maximal monotone and $F \in C(\mathbb{R} \times H, H)$ such that, for some $T > 0$,

$$F(t + T, \cdot) = F(t, \cdot), \quad \forall t \in \mathbb{R}.$$ 
Then we are interested in the existence of $T$-periodic solutions for the semilinear abstract equation:

\begin{equation}
\begin{aligned}
-\Delta u + Au + F(t, u) &= 0 & t \in \mathbb{R} \\
u(0) &= u(T), \quad u'(0) = u'(T)
\end{aligned}
\end{equation}

Let now $H := L^2((0, T); H)$ and $L u := -u'' + Au$, with $D(L) := \{ u \in C^2([0, T]; H) \cap L^2((0, T), D(A)) | u(0) = u(T), \quad u'(0) = u'(T) \}$. $L$ is maximal monotone and if $F$ satisfy (i) and (ii), in particular, a condition of type (13), then problem (15) has exactly one periodic solution.

REFERENCES: