Multivalued Variational Principles and Normed Coercivity

Mihai TURINICI

Abstract. A normed coercivity result is established for a class of order nonsmooth multivalued functionals fulfilling an appropriate Palais-Smale condition. The core of this approach is an asymptotic type statement involving such objects, obtained by means of a multivalued variational principle comparable with the one in Chen, Huang and Hou [J. Optimiz. Th. Appl., 106 (2000), 151-164].

Keywords: Metric/normed space, quasi-order, self-closeness, multivalued variational principle, coercive functional, strong slope, Palais-Smale condition, contingent cone, Dini derivative.


1 Introduction

Let \((X, \| \cdot \|)\) be a (real) Banach space; and \((X^*, \| \cdot \|)\), its topological dual. Given a (proper) functional \(x \mapsto f(x)\) from \(X\) to \(R\) we say that it is coercive, if

\[(a01) \quad f(u) \to \infty, \text{ provided } u \to \infty \text{ (in the sense: } \|u\| \to \infty).\]

Sufficient conditions for deducing such a property involve a differential setting; and the most natural approach is a recursion to the celebrated 1964 Palais-Smale condition [23]. A typical result in this direction is the one due to Caklovic, Li and Willem [7]. Precisely, assume that

\[(a02) \quad f \text{ is Gateaux differentiable and lower semicontinuous (lsc).}\]

Then the property (a01) is deductible (when \(f\) is bounded from below) under a Palais-Smale requirement like

\[(a03) \quad \text{each sequence } (v_n) \text{ in } X \text{ with } (f(v_n)) \text{ bounded and } f'(v_n) \to 0 \text{ (in } X^*) \text{ has a convergent (in } X) \text{ subsequence.}\]

Note that (a02) holds when \(f \in C^1(X)\); hence, their statement includes the one due to Brezis and Nirenberg [6]. An extension of this result (under the same condition (a02)) was obtained by Goeleven [14]. Specifically, the functional considered there is taken as \(f = g + h\), where

\[1\text{This research was supported by Grant PN II PCE ID.387, from the National Authority for Scientific Research, Romania.}\]
(a04) $g$ is Gateaux differentiable lsc and $h$ is (proper) convex lsc;

and the Palais-Smale condition (a03) is adapted to this decomposition. Some enlargements
of this contribution were given in the paper by D. Motreanu and V. V. Motreanu [18]; where
(a04) is to be taken as

(a05) $g$ is locally Lipschitz and $h$ is proper convex lsc

and the Palais-Smale requirement to be used is that in Motreanu and Panagiotopoulos [20, Ch 3]. Further aspects of "functional" nature were considered in Zhong [33] and Turinici [28].

A natural extension of all these facts is to be reached when (the univalued functional)
$f : X \to \mathbb{R}$ is being substituted by a multivalued mapping $x \mapsto F(x)$ from $X$ to $\mathcal{P}_0(\mathbb{R}) := \{Q \subseteq \mathbb{R}; Q \neq \emptyset\}$. The basic result obtained in this direction is the one due to Kristaly and Varga [17]; and consists of two main ingredients:

i) a variational principle involving such maps, due to Chen, Huang and Hou [9], [10]

ii) a Palais-Smale condition based on contingent derivatives taken as in Aubin and Frankowska [3, Ch 4].

It is our aim in this exposition to show that further enlargements of this contribution
are possible; details will be given in Section 4. The basic tool of it is the same variational
principle in i); which, as we shall see, may be viewed as a standard (univalued) one, with
respect to the min-selection $f : X \to \mathbb{R}$ (given as $f(x) = \min(F(x)); x \in X$); details
will be given in Section 3. In addition, it is worth noting that, for deducing this variational
statement, the Brezis-Browder principle [5] will suffice. And, the specific one is a multivalued
version of the strong slope operator (for univalued functionals) introduced by DeGiorgi,
Marino and Tosques [12]. Some particular versions of these developments are discussed in
Section 5. Further aspects of the obtained facts will be delineated elsewhere.

### 2 Brezis-Browder statements

Let $M$ be some nonempty set. Take a quasi-order (i.e.: reflexive transitive relation) $(\leq)$
over $M$; as well as a function $x \mapsto \varphi(x)$ from $M$ to $\mathbb{R}_+ := [0, \infty]$. Call the point $z \in M$, $(\leq, \varphi)$-maximal when: $w \in M$ and $z \leq w$ imply $\varphi(z) = \varphi(w)$. A basic result about the
existence of such points is the 1976 Brezis-Browder ordering principle [5]:

**Proposition 1** Suppose that

(b01) $(M, \leq)$ is sequentially inductive:
   each ascending sequence has an upper bound (modulo $(\leq)$)

(b02) $\varphi$ is $(\leq)$-decreasing ($x \leq y \implies \varphi(x) \geq \varphi(y)$).

Then, for each $u \in M$ there exists a $(\leq, \varphi)$-maximal $v \in M$ with $u \leq v$.

Note that $\varphi(M) \subseteq \mathbb{R}_+$ is not essential for the conclusion above; see Cărjă, Necula and
Vrabie [8, Ch 2, Sect 2.1] for details. Moreover (as established there), Proposition 1 is
reduceable to the Principle of Dependent Choices (see, e.g., Wolk [31]). Finally (cf. Zhu and Li [34]), $(R_+, \geq)$ may be substituted by a separable ordered structure $(P, \leq)$ without altering the conclusion above; see also Turinici [29].

(A) This principle, including Ekeland’s [13], found some useful applications to convex and nonconvex analysis (cf. the above references). For this reason, it was the subject of many extensions; such as the ones in Altman [1], Anisiu [2] and Szaz [25]. These are interesting from a technical viewpoint; but, whenever a maximality principle of this type is to be applied, a substitution of it by the Brezis-Browder’s is always possible. This raises the question of to what extent are these enlargements of Proposition 1 effective. As we shall see, the answer is essentially negative; to do this, some conventions are needed. By a pseudometric over $M$ we shall mean any map $d : M \times M \to R_+$. If, in addition, $d$ is reflexive $[d(x, x) = 0, \forall x \in M]$, triangular $[d(x, z) \leq d(x, y) + d(y, z), \forall x, y, z \in M]$ and symmetric $[d(x, y) = d(y, x), \forall x, y \in M]$, we say that it is a semimetric over $M$. Suppose that we fixed such an object. Call the point $z \in M$, $(\leq, d)$-maximal, in case: $w \in M$ and $z \leq w$ imply $d(z, w) = 0$. Note that, if (in addition) $d$ is sufficient $[d(x, y) = 0$ implies $x = y]$, this property becomes: $w \in M, z \leq w \implies z = w$ (and reads: $z$ is strongly $(\leq)$-maximal).

So, existence results involving such points may be viewed as “metrical” versions of the Zorn-Bourbaki maximality principle [35], [4]. To get sufficient conditions for these, one may proceed as below. Let $(x_n)$ be an ascending sequence in $M$. The $d$-Cauchy property for it is introduced in the usual way: $\forall \varepsilon > 0, \exists n(\varepsilon)$ such that $n(\varepsilon) \leq p \leq q \implies d(x_p, x_q) \leq \varepsilon$. Also, call this sequence $d$-asymptotic, when $d(x_n, x_{n+1}) \to 0$, as $n \to \infty$. Clearly, each (ascending) $d$-Cauchy sequence is $d$-asymptotic too. The reverse implication is also true when all such sequences are involved; i.e., the global conditions below are equivalent:

(b03) each ascending sequence is $d$-Cauchy

(b04) each ascending sequence is $d$-asymptotic.

By definition, either of these will be referred to as $(M, \leq)$ is regular (modulo $d$). Moreover, this property implies its relaxed version

(b05) $(M, \leq)$ is weakly regular (modulo $d$): $\forall x \in M, \forall \varepsilon > 0, \exists y = y(x, \varepsilon) \geq x$ such that $y \leq u \leq v \implies d(u, v) \leq \varepsilon$.

The following ordering principle is available (cf. Kang and Park [16]):

**Proposition 2** Assume that (b01) and (b05) are true. Then, for each $u \in M$ there exists a $(\leq, d)$-maximal $v \in M$ with $u \leq v$.

As a direct consequence of this we have (cf. Turinici [26]):

**Proposition 3** Assume that $(M, \leq)$ is sequentially inductive and regular (modulo $d$). Then, the conclusion of Proposition 2 is retainable.

Now (see the above reference) Prop 1 $\implies$ Prop 2. On the other hand, Prop 2 $\implies$ Prop 3 in a trivial way. Finally, Prop 3 $\implies$ Prop 1: just take $d(x, y) = |\varphi(x) - \varphi(y)|, x, y \in M$ (where $\varphi$ is the above one). Summing up, all these variants of the Brezis-Browder ordering principle (Proposition 1) are nothing but logical equivalents of it.
(B) A basic application of these facts is to "monotone" variational principles. Let \( M \) be a nonempty set. Take a quasi-order \((\preceq)\) and a metric \( d : M \times M \to \RR_+ \) over it; the resulting triple will be termed a quasi-ordered metric space. Call the subset \( Z \) of \( M \), \((\preceq)\)-closed when the limit of each ascending (modulo \((\preceq)\)) sequence in \( Z \) belongs to \( Z \). Clearly, any closed part of \( M \) is \((\preceq)\)-closed too; but the converse is not in general true. (Just take \( M = \RR \) (endowed with the usual order/metric); and \( Z = \{0, 1\} \)). Further, call the quasi-order \((\preceq)\), self-closed provided \( M(x, \preceq) := \{ u \in M; x \preceq u \} \) is \((\preceq)\)-closed, for each \( x \in M \); or, equivalently: the limit of each ascending sequence is an upper bound of it (modulo \((\preceq)\)).

Finally, call the ambient metric \( d \), \((\preceq)\)-complete provided each ascending (modulo \((\preceq)\)) \( d\)-Cauchy sequence converges. As before, if \( d \) is complete, then it is \((\preceq)\)-complete too. The reciprocal is not in general true; take \( M = \{0, 1\} \) endowed with the standard order/metric.

We may now state the announced result. Take a function \( \varphi : M \to \RR \cup \{\infty\} \) fulfilling

\[
\text{(b06)} \quad \varphi \text{ is inf-proper (Dom}(\varphi) \neq \emptyset \text{ and } \varphi_* := \inf\{\varphi(M)\} > -\infty)
\]

\[
\text{(b07)} \quad \varphi \text{ is \((\preceq)\)-lsc over } M: [\varphi \leq t := \{ x \in X; \varphi(x) \leq t \} \text{ is \((\preceq)\)-closed, } \forall t \in \RR.
\]

**Proposition 4** Let \((\preceq)\) be self-closed and \( d \) be \((\preceq)\)-complete. Then

i) for each \( u \in \text{Dom}(\varphi) \) there exists \( v \in \text{Dom}(\varphi) \) with

\[
u \leq v, d(u, v) \leq \varphi(u) - \varphi(v) \text{ (hence } \varphi(u) \geq \varphi(v))
\]

\[
d(v, x) > \varphi(v) - \varphi(x), \text{ for each } x \in M(v, \preceq) \setminus \{v\}.
\]

ii) if \( u \in \text{Dom}(\varphi) \), \( \rho > 0 \) fulfill \( \varphi(u) - \varphi_* \leq \rho \), then \((2.1)\) gives

\[
\varphi(u) \geq \varphi(v) \text{ and } u \leq v, d(u, v) \leq \rho.
\]

The original argument is that appearing in Turinici [27]. For the sake of completeness, we shall provide it, with some modifications.

**Proof (of Proposition 4)** Denote for simplicity \( M[u] = \{ x \in M; u \preceq x, \varphi(u) \geq \varphi(x) \} \). Clearly, \( \emptyset \neq M[u] \subseteq \text{Dom}(\varphi) \); moreover, by \((b07)\) (and the choice of \((\preceq)\))

\[
M[u] \text{ is \((\preceq)\)-closed; hence } d \text{ is \((\preceq)\)-complete on } M[u].
\]

Let \((\preceq)\) stand for the relation (over \( M \)): \( x \preceq y \text{ iff } x \preceq y, d(x, y) + \varphi(y) \leq \varphi(x) \). It is not hard to see that \((\preceq)\) acts as and order (antisymmetric quasi-order) on \( \text{Dom}(\varphi) \); so, it remains as such on \( M[u] \). We claim that conditions of Proposition 3 are fulfilled on \((M[u]; \preceq; d)\). In fact, let \((x_n)\) be an ascending (modulo \((\preceq)\)) sequence in \( M[u] \):

\[
\text{(b08)} \quad x_n \leq x_m \text{ and } d(x_n, x_m) \leq \varphi(x_n) - \varphi(x_m), \text{ if } n \leq m.
\]

The sequence \((\varphi(x_n))\) is descending and (by \((b06)\)) bounded from below; hence a Cauchy one. This, along with the preceding relation, shows that \((x_n)\) is an ascending (modulo \((\preceq)\)) \( d\)-Cauchy sequence; wherefrom \((M[u], \preceq)\) is regular (modulo \( d \)). Moreover, the obtained properties give us (by \((2.4)\)) some \( y \in M[u] \) with \( x_n \to y \). Combining with \((b08)\) one derives (via \((b07)\) and the choice of \((\preceq)\))

\[
x_n \leq y, d(x_n, y) \leq \varphi(x_n) - \varphi(y), \text{ (i.e.: } x_n \preceq y, \text{ for all } n.
\]
In other words, \( y \in M[u] \) is an upper bound (modulo \((\leq)\)) of \((x_n)\); and this shows that \((M[u], \leq)\) is sequentially inductive. By Proposition 3 it then follows that, for the starting \( u \in M[u] \) there exists \( v \in M[u] \) with \( j \) \( u \preceq v \) and \( j \) \( v \) is \((\leq, d)\)-maximal in \( M[u] \). The former of these is just (2.1). And the latter one gives at once (2.2); because it reads: \( x \in M[u] \) and \( v \preceq x \) imply \( v = x \). The last part is evident; so, the conclusion follows. \( \Box \)

A basic particular case of our developments corresponds to the choice \((\leq) = M \times M\) (the trivial quasi-order on \( M \)). The regularity condition (b07) may then be written as

\[(b09) \quad \varphi \text{ is lsc over } M: \liminf_n \varphi(x_n) \geq \varphi(x), \text{ whenever } x_n \to x;\]

and Proposition 4 is nothing but Ekeland’s variational principle [13] (in short: EVP). On the other hand, the same requirement holds under (b02) and the self-closeness of \((\leq)\). For this reason, Proposition 4 will be called the monotone version of EVP. Note that, by the remarks above, it may be derived from Proposition 1 as well; we do not give details. Further aspects may be found in Hyers, Isac and Rassias [15, Ch 5].

## 3 Multivalued variational principles

Let \((X, d)\) be a complete metric space; and \( F : X \to \mathcal{P}_0(R) \) be a multivalued map from \( X \) to \( R \). As usually, we shall identify \( F \) with its graph in \( X \times R \); so, \( F \) is a relation between \( X \) and \( R \) with \( \text{Dom}(F) = X \). Define a quasi-order \((\preceq)\) over \( X \times R \) as

\[(c01) \quad (x, t) \preceq (y, s) \text{ iff } d(x, y) \leq t - s.\]

Clearly, it is in addition antisymmetric; hence a (partial) order. The question to be posed is the following: under which conditions about our data is \((\leq)\), admissible over \( F \) in the Zorn-Bourbaki sense [each point of \( F \) is majorized by a \((\leq)\)-maximal one]. For an appropriate answer, note that, in the univalued case, these are (c) the boundedness from below and (cc) the lsc property (see, for instance, Ekeland [13]). It is our aim to show that this is formally retainable in our multivalued case too. Precisely, assume that

\[(c02) \quad F \text{ is bounded from below: } F_* := \inf[F(X)] > -\infty\]

\[(c03) \quad F \text{ is submonotone: for each } ((x_n, t_n)) \subseteq F \text{ with } x_n \to x \text{ and } (t_n) \text{ descending,}\]

\[\text{there exists } t \in F(x) \text{ with } t_n \geq t, \text{ for all } n.\]

**Theorem 1** Let these conditions hold. Then, for each \((x_0, t_0) \in F\) there exists \((\bar{x}, t) \in F\) with i) \((x_0, t_0) \preceq (\bar{x}, t)\), ii) \((\bar{x}, t) \preceq (x, t) \in F \implies (\bar{x}, t) = (x, t)\) and iii) \( t = \inf[F(\bar{x})] \).

**Proof** Let \( e \) stand for the "product" metric: \( e((x_1, t_1), (x_2, t_2)) = d(x_1, x_2) + |t_1 - t_2|\), \((x_1, t_1), (x_2, t_2) \in X \times R \). We claim that Proposition 3 is applicable to \((F; \leq; e)\); and this will complete the argument. Let \(((x_n, t_n)) \subseteq F\) be a \((\leq)\)-ascending sequence; i.e.,

\[(c04) \quad d(x_n, x_m) \leq t_n - t_m, \text{ whenever } n \leq m.\]
The sequence \((t_n)\) is descending and bounded from below in \(F(X)\) (by (c02)); hence, a Cauchy one. This, along with (c04), shows that \((x_n)\) is \(d\)-Cauchy in \(X = \text{Dom}(F)\); and so, \((F, \preceq)\) is regular (modulo \(e\)). As \((X, d)\) is complete, \(x_n \to x\) for some \(x \in X\); so (combining with (c03) above) there must be a \(t \in F(x)\) with \(t_n \geq t, \forall n\). By (c04),

\[
(\forall n) : \quad d(x_n, x) \leq d(x_n, x_m) + d(x_m, x) \leq t_n - t + d(x_m, x), \forall m \geq n.
\]

Passing to limit as \(m \to \infty\) gives \(d(x_n, x) \leq t_n - t\) (i.e.: \((x_n, t_n) \preceq (x, t)\)), \(\forall n\); which tells us that \((x, t) \in F\) is an upper bound of \(((x_n, t_n))\); i.e., \((F, \preceq)\) is sequentially inductive; hence the claim. By Proposition 3 it then follows that, for the starting \((x_0, t_0) \in F\) there exists a \((x, t) \in F\) with the properties i) and ii) in the statement. Finally, note that (by (c02)) \(-\infty < \inf[\{F(\bar{x})\}] \leq t\). Assume by absurd that \(t > \inf[\{F(\bar{x})\}]\). By definition, there must be some \(t^* \in F(\bar{x})\) with \(t > t^*\). But then, \((\bar{x}, t) \preceq (\bar{x}, t^*)\), \((\bar{x}, t) \not\preceq (\bar{x}, t^*)\); in contradiction with the maximality of \((\bar{x}, t)\) in \((F, \preceq)\). Hence, iii) holds as well; and conclusion follows. \(\square\)

Note that (c03) holds (under (c02)) when

(c05) \(F\) is closed: \(((x_n, t_n)) \subseteq F, x_n \to x, t_n \to t\) imply \((x, t) \in F\).

For, if \(((x_n, t_n))\) is as in the premise of (c03), \(t = \lim_n(t_n)\) exists; and then \(t \in F(x)\) by (c05). In this case, the corresponding version of Theorem 1 is compatible with the related statement in Phelps [24, Ch 3, Sect 3.12]. So, a discussion of (c05) would be useful. This will necessitate some conventions. We say that \(F\) is upper semicontinuous at \(x \in X\) if

(c06) \(\forall \varepsilon > 0, \exists \delta > 0: x \in X(x, \delta) \implies F(x) \subseteq R(F(x), \varepsilon)\).

Here, for each metric space \((Z, e)\) and each couple \((W \subseteq Z; \gamma > 0)\) we put \(Z(W; \gamma) = \{z \in Z; \text{dist}(z, W) < \gamma\}\); where "dist" is the associated point to set distance in \(Z\). We now claim that (c05) is obtainable from

(c07) \(F\) is (nonempty) closed valued: \((\emptyset \neq) F(x) = \text{closed, } \forall x \in X\)

(c08) \(F\) is upper semicontinuous on \(X\) (i.e.: (c06) holds, for each \(x \in X\)).

In fact, let \(((x_n, t_n)) \subseteq F\) be as in the premise of this condition. Given the arbitrary fixed \(\varepsilon > 0\), let \(\delta > 0\) be the number appearing in (c08). As \(x_n \to x\), there must be some \(n(\delta)\) in such a way that \(n > n(\delta)\) implies \(d(x_n, x) < \delta\). This, along with (c06), yields (for all such \(n\)) \(F(x_n) \subseteq R(F(x), \varepsilon)\); hence \(t_n \in R(F(x), \varepsilon), \forall n \geq n(\delta)\). Passing to limit as \(n \to \infty\) gives \(\text{dist}(t, F(x)) \leq \varepsilon\); so that (by the arbitrariness of \(\varepsilon > 0\) and (c07)) \(t \in F(x)\); hence the claim.

(B) An interesting particular case of these developments corresponds to \(F\) being univalued. So, let \(f : X \to R\) be some function with

(c09) \(f\) is bounded from below (\(f_* := \inf[f(X)] > -\infty\))

(c10) \(f\) is submonotone: for each \((x_n) \subseteq X\) with \(x_n \to x\) and \((f(x_n))\) descending, we have \(f(x_n) \geq f(x), \forall n\).

Theorem 2 Let these conditions hold. Then, for each \(x_0 \in X\) there exists \(\bar{x} \in X\) with \(j\)

\[
d(x_0, \bar{x}) \leq f(x_0) - f(\bar{x}), \forall x \in X, d(\bar{x}, x) \leq f(\bar{x}) - f(x) \implies \bar{x} = x.
\]
Note that (c10) is just the descending-lsc property used by Nemeth [22]. In fact, Theorem 2 gives an extended version of Ekeland’s variational principle [13] (in short: EVP) based on (c09)+(c10). For, let \( f : X \to \mathbb{R} \cup \{\infty\} \) with \( \text{Dom}(f) \neq \emptyset \) be taken as precise; and \( x_0 \in \text{Dom}(f) \) be arbitrary fixed. Then, an application of Theorem 2 to \( X_0 := \{ x \in X; f(x) \leq f(x_0) \} \) gives the conclusions \( j \) and \( jj \) above; hence the claim. Clearly, this is also true for the standard form of EVP, with (c10) substituted by its stronger counterpart (b09) (modulo \( f \)). A natural question to be posed is that of the reciprocal implication (c10) \( \iff \) (b09) being available. The answer to this is negative; as shown by

**Example 1** Take \( X = R \) (endowed with the usual order/metric); and define

\[
(f : R \to R) \quad f(t) = c^t - 1, \text{ if } t < 0; \quad f(t) = t + 2, \text{ if } t \geq 0.
\]

Clearly, (b09) does not hold for this object; because \( \{ t \in R; f(t) \leq 1 \} = -\infty, 0 \} \) is not closed. On the other hand, (c10) is retainable for \( f \). In fact, let \( (t_n) \in R \) and \( t \in R \) be such that \( t_n \to t \) and \( (f(t_n)) \) is descending. As \( f \) is strictly increasing, \( (t_n) \) is descending too. As a consequence, \( t_n \geq t \), for all \( n \); wherefrom \( f(t_n) \geq f(t) \), for all \( n \); hence the claim.

(C) The above developments raise the question of to what extent is Theorem 1 deductible from Theorem 2. Let \( F : X \to P_0(R) \) be such that \( (c02)+(c03) \) hold. Call the subset \( P \subseteq R \), inf-closed provided \( -\infty < \inf(P) \in P \).

**Lemma 1** Under these conditions, we have

\[
F(x) \text{ is inf-closed, for each } x \in X. \tag{3.1}
\]

**Proof** By (c02), we have for the moment \( \inf[F(x)] > -\infty \). Let \( (t_n) \subseteq F(x) \) be descending and \( \lim_n(t_n) = \inf[F(x)] \). By (c03) applied to the sequence \( (x, t_n) \) \( \subseteq F \) there exists \( t \in F(x) \) with \( t_n \geq t \), for all \( n \). But then, \( t_0 = \inf[F(x)] \); hence the conclusion. \( \square \)

As a consequence, the function \( f : X \to R \) given by

\[
(c11) \quad f(x) = \inf[F(x)], \quad x \in X \text{ (in short: } f = \inf[F])
\]

is well defined. Moreover, by (3.1), \( f \) is a selection of \( F \) [i.e.: \( f(x) \in F(x), \forall x \in X \)] with: \( f \leq g \), for any other selection \( g \) of \( F \). The question to be posed is: which properties of \( F \) are transferable to \( f \). A partial answer to this is contained in

**Lemma 2** Let the precise conditions \( (c02)+(c03) \) hold. Then, \( f := \inf[F] \) is bounded below and submonotone (in the sense of \( (c09)+(c10)) \).

**Proof** Let \( (x_n) \subseteq X \) be such that \( x_n \to x \) and \( (f(x_n)) \) is descending. By (c03), there must be \( t \in F(x) \) such that \( f(x_n) \geq t \), for all \( n \). On the other hand (by definition) \( t \geq f(x) \). Combining these gives the conclusion. \( \square \)

We are now in position to give an appropriate answer to the question above.

**Proposition 5** Under the precise conditions, we have \( \text{Theorem 2} \implies \text{Theorem 1}; \) hence \( \text{Theorem 2} \iff \text{Theorem 1} \).
Proof Let the premises of Theorem 1 be accepted; and \((x_0, t_0) \in F\) be arbitrary fixed. By Lemma 2, conditions (c09)+(c10) are valid for \(f := \inf[F]\). Hence, Theorem 2 applies here; so that (by the notations we already introduced) for the starting \((x_0, f(x_0)) \in F\) there exists \((\hat{x}, f(\hat{x})) \in F\) with \(h) (x_0, t_0) \preceq (\hat{x}, f(\hat{x}))\) and \(hh) (\hat{x}, f(\hat{x})) \preceq (x, f(x)) \implies \hat{x} = x\). Combining with \((x_0, t_0) \preceq (x_0, f(x_0))\) yields i) with \(t = f(\hat{x})\). Moreover, iii) holds as well with such a choice; so, it remains to verify ii). Let \((x, t) \in F\) be such that \((\tilde{x}, f(\tilde{x})) \preceq (x, t)\). Since \((x, t) \preceq (x, f(x))\), we get \((\tilde{x}, f(\tilde{x})) \preceq (x, f(x))\); wherefrom (by \(hh)\) above) \(\tilde{x} = x\) (hence \(f(\tilde{x}) = f(x)\)). This finally combined with \(f(\tilde{x}) \geq t \geq f(x)\) yields \(f(\tilde{x}) = t\); hence \((\tilde{x}, f(\tilde{x})) = (x, t)\); and the conclusion follows. 

Summing up, the multivalued versions of such maximal principles are non-effective. So, genuine extensions are possible in a vectorial context; we shall discuss them elsewhere.

4 Normed coercivity

With this information at hand, we may now return to the questions of the introductory part. Let \((X, \|, |\|)\) be a (real) Banach space. Denote by \(d\) the (standard) metric induced by \(\|, |\|\); hence, it is invariant to translations and \((X, d)\) is complete. Further, take some map \(\Gamma: X \to P_0(R)\) with the properties

\[(d01) \quad \Gamma\) is almost \((\lambda, \mu)\)-Lipschitz \((d(x, y) \leq \lambda \implies |\Gamma(x) - \Gamma(y)| \leq \mu)\)

for certain \(\lambda, \mu > 0\) with \(\lambda < 1 < \mu\)

\[(d02) \quad \Gamma(X) has arbitrarily large points: \sup[\Gamma(X)] = \infty.\]

A useful consequence involving the level sets \(\{[\Gamma \geq \sigma] := \{x \in X; \Gamma(x) \geq \sigma\}, \sigma > 0\}\) is

\[\text{cl}(\{[\Gamma \geq \rho]\}) \subseteq X([\Gamma \geq \rho], \lambda) \subseteq [\Gamma \geq \rho - \mu], \forall \rho \geq \mu; \quad (4.1)\]

where, “\(\text{cl}\)” is the closure operator. In fact, let \(v \in X([\Gamma \geq \rho], \lambda)\) be arbitrary fixed. By definition, there must be some \(u \in [\Gamma \geq \rho]\) with \(d(u, v) < \lambda\); hence \(|\Gamma(u) - \Gamma(v)| \leq \mu\) (if we take \((d01)\) into account). But then, \(\Gamma(v) \geq \Gamma(u) - \mu \geq \rho - \mu\) (i.e. \(v \in [\Gamma \geq \rho - \mu]\)); and the claim follows. [Notice that all these level sets are nonvoid, by \((d02)\); wherefrom, the reasoning above is effective]. Finally, pick some multivalued functional \(F: X \to P_0(R)\) with

\[(d03) \quad F is bounded below and submonotone (cf. \((c02)\)\) (c03)).\]

Remember that, as a consequence of this, the (univalued) functional \(f = \inf[F]\) (from \(X\) to \(R\)) is well defined (via \((c11)) as a selection of \(F\) (minimal with respect to this property). In addition (see above)

\[f is bounded below and submonotone (in the sense of \((c09)\)\)\) (c10)).\]

By these remarks, the quantities \(m(\Gamma, f)(\sigma) := \inf[f([\Gamma \geq \sigma])]\) exist in \(R\) for each \(\sigma > 0\). Moreover, the map \(\sigma \mapsto m(\Gamma, f)(\sigma)\) is increasing from \(R^+_\infty := [0, \infty[\) to \(R\); wherefrom

\[\liminf_{\Gamma(u) \to \infty} f(u) := \sup m(\Gamma, f)(\sigma) \geq \lim_{\sigma \to \infty} m(\Gamma, f)(\sigma)\]

(4.2)
exists, as an element of $R \cup \{\infty\}$, in view of
\begin{equation}
 f_* \leq m(\Gamma, f)(\sigma) \leq \alpha(\Gamma, f) := \liminf_{\Gamma(u) \to \infty} f(u) \leq \infty, \quad \forall \sigma > 0. \tag{4.3}
\end{equation}

When $\alpha(\Gamma, f) = \infty$, the functional $f$ will be referred to as \Gamma-coercive; and the same convention applies to $F$. It is our aim in the following to get sufficient conditions in order that such a property be attained. These, as a rule, require a differential setting relative to $f$ and $F$. Denote, for each $u \in X$,\begin{equation}
|\nabla| f(u) = \max \{0, \nabla f(u) := \limsup_{x \to u} \frac{f(u) - f(x)}{d(u, x)} \}. \tag{d04}
\end{equation}

This object is comparable with the one introduced by DeGiorgi, Marino and Tosques [12]; and will be referred to as the strong $d$-slope of $f$ at $u$. The usefulness of such concepts for the critical point theory (for univalued functionals) was underlined by Corvellec, DeGiovanni and Marzocchi [11]. Here, we shall establish that a certain multivalued version of it is the natural tool for our “multivalued” coercivity theory as well. Denote, for $(u, v) \in F$,
\begin{equation}
|\nabla| F(u, v) = \max \{0, \nabla F(u) := \limsup_{x \to u} \frac{v - F(x)}{d(u, x)} \}. \tag{d05}
\end{equation}

As before, we shall term this quantity, the strong $d$-slope of $F$ at $(u, v)$. The connection between these two concepts is discussed in the lemma below. For the subset $P$ of $R$ and the point $r \in R$, put $P \geq r$ whenever $t \geq r$, for each $t \in P$.

**Lemma 3** Under the precise conditions we have, for each $u \in X$,
\begin{equation}
\nabla F(u, f(u)) \leq \nabla f(u); \text{ hence } |\nabla| F(u, f(u)) \leq |\nabla| f(u). \tag{4.4}
\end{equation}

**Proof** Let $\varepsilon > 0$ be arbitrary fixed. By the very definition of $f$,
\begin{equation}
\sup_{d(u, x) < \varepsilon} \frac{f(u) - F(x)}{d(u, x)} \leq \sup_{d(u, x) < \varepsilon} \frac{f(u) - f(x)}{d(u, x)}; \text{ as } F(x) \geq f(x), \text{ for all such } x. \tag{d06}
\end{equation}

Passing to infimum (=limit) as $\varepsilon \to 0$ yields the needed conclusion. \hfill \Box

The following asymptotic type statement is a basic step to the answer we are looking for.

**Theorem 3** Suppose that
\begin{equation}
(\text{d06}) \quad \alpha(\Gamma, f) < \infty \quad (\text{hence (cf. (4.3)) } \alpha(\Gamma, f) \text{ is finite}). \tag{4.5}
\end{equation}

There exists then, a sequence $(v_n)$ in $\Gamma^{-1}(R^0_+)$ with
\begin{align*}
\Gamma(v_n) &\to \infty \text{ (so, } \Gamma(y_n) \to \infty \text{ for each subsequence } (y_n) \text{ of } (v_n)) \tag{4.5} \\
f(v_n) &\to \alpha(\Gamma, F) \quad \text{as } n \to \infty \tag{4.6} \\
|\nabla| f(v_n) &\to 0 \quad (\text{hence } |\nabla| F(v_n, f(v_n)) \to 0). \tag{4.7}
\end{align*}
Proof There are two steps to be passed.

(I) Let the parameter $\eta > 0$ be taken according to

\[(d07) \eta < \frac{\lambda}{2\mu}; \text{ hence (according to (d01)) } \frac{1}{\eta} > \mu > \frac{\lambda}{2} > \eta.\]

By (d06), there exists $r(\eta)$ with

\[r(\eta) \geq \frac{1}{\eta}; \quad m(\Gamma, f)(r) > \alpha(\Gamma, f) - \eta^2, \forall r \geq r(\eta). \quad (4.8)\]

Having this precise, we claim that there exists $v_\eta \in X$ so that

\[\Gamma(v_\eta) > r(\eta), |f(v_\eta) - \alpha(\Gamma, f)| < \eta^2,\]

\[|\nabla|F(v_\eta, f(v_\eta))| \leq |\nabla|f(v_\eta) \leq \eta.\]

\[ (4.9)\]

\[|\nabla|F(v_\eta, f(v_\eta))| \leq |\nabla|f(v_\eta) \leq \eta.\]

\[ (4.10)\]

We claim that $v_\eta$ is our desired point. In fact, (4.1) and the definition of $M$ give (by (d07) and (4.8) (the first half))

\[v_\eta \in [\Gamma \geq 2r(\eta) - \mu] \subset [\Gamma \geq r(\eta)];\]

\[ (4.13)\]

and, from this, the first part of (4.9) is clear. Combining with the second half of (4.8) and (4.11) gives

\[\eta d(u_\eta, v_\eta) \leq f(u_\eta) - f(v_\eta) \leq f(u_\eta) \geq f(v_\eta);\]

\[ (4.11)\]

so, the second part of (4.9) holds too. This, again coupled with (4.11) yields (via (d07))

\[d(u_\eta, v_\eta) \leq (1/\eta)2d(u_\eta, \lambda) < \lambda; \text{ wherefrom, by (4.1), }\]

\[v_\eta \in X(u_\eta, \lambda) \subset [\Gamma \geq 4r(\eta) - \mu];\]

\[ (4.15)\]

which ”improves” (4.13) above. Finally, again by (4.1) (and (d07)),

\[X(v_\eta, \lambda) \subset [\Gamma \geq 4r(\eta) - 2\mu] \subset [\Gamma \geq 2r(\eta)] \subset M.\]

Summing up, $v_\eta$ is an interior point of $M$ fulfilling the variational condition (4.12). This, along with the definition of the conical strong $d$-slope, gives (4.10); and the claim follows.

(II) Let $(\eta_n)$ be a descending to zero sequence in $[0, \lambda/2\mu]$ and put $r_n = r(\eta_n) =$ the quantity of (4.8), $n \geq 0$. Note that, by this choice, $r_n \geq 1/\eta_n$, for all $n$; hence $r_n \to \infty$.
as \( n \to \infty \). Moreover, the developments in (I) give us a sequence \((v_n = v_{\eta_n})\) in \( \Gamma^{-1}(R^0_+) \) fulfilling

\[
\Gamma(v_n) \geq r_n, \quad |f(v_n) - \alpha(\Gamma, f)| < \eta_n^2, \quad |\nabla|F(v_n, f(v_n))| \leq |\nabla|f(v_n)| \leq \eta_n, \quad \forall n. \quad (4.16)
\]

But, from this, (4.5)-(4.7) are clear. The proof is thereby complete. \( \square \)

We are now in position to give the promised answer to our coercivity question. The "hybrid" condition below is to be considered

\[
\text{(d08)} \quad \text{each sequence } (x_n) \text{ in } \Gamma^{-1}(R^0_+) \text{ for which } (f(x_n)) \text{ converges and } |\nabla|F(x_n, f(x_n))| \to 0
\]

has a subsequence \((y_n)\) with \((\Gamma(y_n))\) bounded (in \( R_+ \)).

This will be referred to as a Palais-Smale condition upon \( f \) with respect to \( \Gamma \).

**Theorem 4** Suppose that (in addition) \( f \) satisfies a Palais-Smale condition with respect to \( \Gamma \). Then, \( f \) (hence \( F \) as well) is \( \Gamma \)-coercive.

**Proof** If, by absurd, this cannot happen, the relation (d06) must be true. By Theorem 3, we have promised a sequence \((v_n)\) in \( \Gamma^{-1}(R^0_+) \) with the properties (4.5)-(4.7). Combining with the imposed Palais-Smale condition it results that \((v_n)\) must have a subsequence \((y_n)\) with \((\Gamma(y_n))\) bounded (in \( R_+ \)). On the other hand, \( \Gamma(y_n) \to \infty \), by (4.5). The obtained contradiction shows that (d06) cannot be accepted; hence the conclusion. \( \square \)

In particular, (d08) follows (via Lemma 3) from

\[
\text{(d09)} \quad \text{each sequence } (x_n) \text{ in } \Gamma^{-1}(R^0_+) \text{ for which } (f(x_n)) \text{ converges and } |\nabla|F(x_n, f(x_n))| \to 0
\]

has a subsequence \((y_n)\) with \((\Gamma(y_n))\) bounded (in \( R_+ \));

this will be referred to as the Palais-Smale condition upon \( F \) with respect to \((\Gamma, f)\). As a consequence, we have

**Theorem 5** Suppose that (in addition) \( F \) satisfies a Palais-Smale condition with respect to \((\Gamma, f)\). Then, \( F \) is necessarily \( \Gamma \)-coercive.

Summing up, these "multivalued" coercivity results reduce to their "univalued" versions. So, genuine extensions of this type are to be obtained in a vectorial setting. On the other hand, a (quasi-) order extension of this result is available under the lines in D. Motreanu, V. V. Motreanu and M. Turinici [19]. We shall discuss all these in a future paper.

## 5 Differential versions

The obtained results are, at a first glance, "absolute" ones. For technical reasons, it would be useful having "relative" forms of them (expressed via Dini derivatives).

**A.** Let \((X, ||.||)\) be a normed space; and \( K \) be some nonempty part of \( X \). The contingent cone of \( K \) at some \( z \in K \) is defined as

\[
\text{(e01)} \quad T(K)(z) = \{ w \in X; \liminf_{\lambda \to 0} (1/\lambda) \text{ dist}(z + \lambda w, K) = 0 \};
\]
here, as already precise, dist(.,.) is the point to set distance attached to d (=the metric induced by ||.,||). Note that \( w \in \mathcal{T}(K)(z) \) if and only if
\[
\exists(\lambda_n) \subseteq R^0_+, \lambda_n \to 0, \exists(z_n) \subseteq K : w = \lim_n(1/\lambda_n)(z_n - z).
\]
This also writes, for simplicity reasons
\[(e02) \ w \in \text{Lim}_n(1/\lambda_n)(K - z), \text{ for some sequence } (\lambda_n) \subseteq R^0_+, \lambda_n \to 0.
\]
Note that the effective meaning of this is
\[
\liminf_{\lambda \to 0^+} \text{dist}[w,(1/\lambda)(K - z)] = 0; \text{ and writes: } w \in \text{Lim sup } (1/\lambda)(K - z);
\]
cf. Zălinescu [32, Ch 3, Sect 3.1]. Here, for each metric space \((Z,e)\) and each multivalued map \(G : Z \to \mathcal{P}_0(X)\) with \(\text{Dom}(G) = Z\) we denoted
\[(e03) \ \text{Lim sup } G(z) = \{w \in X; \text{lim inf } \text{dist}(w,G(z)) = 0\}, \ c \in Z.
\]
Note that the conical property of \(\mathcal{T}(K)(z)\) must be taken in the homogeneous sense only:
\[
w \in \mathcal{T}(K)(z) \implies \alpha w \in \mathcal{T}(K)(z), \forall \alpha > 0. \tag{5.1}
\]
For, in general, \(\mathcal{T}(K)(z)\) is not convex unless \(K\) is convex. Nevertheless, \(\mathcal{T}(K)(z)\) is anyway closed; see Ward [30] for a thorough discussion of these facts.

(B) Now, let \((X,||.||), (Y,||.||)\) be a couple of normed spaces; and \(F : X \to \mathcal{P}_0(Y)\) be a multivalued map from \(X\) to \(Y\). As usually, we identify \(F\) with its graph in \(X \times Y\). The Dini derivative of \(F\) at \((x,y) \in F\) is the multivalued map from \(X\) to \(Y\):
\[(e04) \ \Delta F(x,y) = \{(u,v) \in X \times Y; (u, -v) \in \mathcal{T}(F)(x,y)\}.
\]
By the characterization of the contingent cone in the right hand side, it follows that \((u,v) \in \Delta F(x,y)\) if and only if
\[
\exists(\lambda_n) \subseteq R^0_+, \lambda_n \to 0, \exists((x_n,y_n)) \subseteq F : u = \lim_n(1/\lambda_n)(x_n - x), v = \lim_n(1/\lambda_n)(y - y_n).
\]
Putting \(w_n = (1/\lambda_n)(x_n - x)\), we have \(x_n = x + \lambda_nw_n\) (for all \(n\)); whence
\[
(y_n \in F(x_n) = F(x + \lambda_nw_n), n \geq 0); \text{ with } \lambda_n \to 0, w_n \to u.
\]
So, for a fixed \(u \in X\), one has \(v \in \Delta F(x,y)(u)\) if and only if
\[(e05) \ \exists(\lambda_n) \subseteq R^0_+ , \lambda_n \to 0, \exists(w_n) \subseteq X w_n \to u, \exists(y_n) \subseteq Y:
\]
\[
(y_n \in F(x + \lambda_nw_n); n \geq 0) \text{ and } v = \lim_n(1/\lambda_n)(y - y_n).
\]
This also writes, for simplicity
\[(e06) \ \exists(\lambda_n) \subseteq R^0_+, \lambda_n \to 0, \exists(w_n) \subseteq X w_n \to u: v \in \text{Lim}_n(1/\lambda_n)(y - F(x + \lambda_nw_n));
\]
or equivalently (by the definition above)
\[ v \in \text{Lim sup} \left( 1/\lambda \right) (y - F(x + \lambda w)); \]

see, for instance, Aubin and Frankowska [3, Ch 5, Sect 5.1]. The multivalued map \( u \mapsto \Delta F(x,y)(u) \) exists at least in \( u = 0 \); precisely,
\[ (0 \in \text{Dom}(\Delta F(x,y)) \text{ and } 0 \in \Delta F(x,y)(0)). \]

Moreover, it is positively homogeneous over its domain
\[ \Delta F(x,y)(\alpha u) = \alpha \Delta F(x,y)(u), \forall \alpha > 0; \]
where, by convention, \( \alpha \emptyset = \emptyset, \forall \alpha > 0 \). Unfortunately, \( \text{Dom}(\Delta F(x,y)) = \{0\} \) cannot be avoided; i.e.: for many nonzero directions \( u \in X \) the limit in (e06) may be empty.

(C) Let \( F : X \to \mathcal{P}_0(R) \) be a multivalued map from \( X \) to \( R \). Define, for \( (x,y) \in F \)
\[ \Delta_x \sup_{\mathcal{P}_0(R)} F(x,y)(u) = \sup[\Delta F(x,y)(u), u \in X]. \]

As usually, the supremum of the empty set is \((-\infty)]\). The obtained map \( u \mapsto \Delta_x \sup_{\mathcal{P}_0(R)} F(x,y)(u) \) is well defined over all of \( X \), with values in \( \tilde{R} := R \cup \{ -\infty, \infty \} \). Hence, the convention below is meaningful, for each \( (x,y) \in F \):

\[ \exists (\delta F(x,y)(\alpha u) = \sup\{\Delta_x \sup_{\mathcal{P}_0(R)} F(x,y)(u); u \in X_x(1)\}). \]

(Here, \( X_x(1) := \{ x \in X; ||x|| = 1 \} \) is the boundary of the unit sphere in \( X \). It will be referred to as the differential strong \( d \)-slope of \( F \) at \((x,y)\). A natural question is to establish the connection between this indicator and the one introduced in Section 4. The following answer is to be noted.

**Lemma 4** Let \((x,y) \in F \) be arbitrary fixed. Then
\[ \delta F(x,y) \leq \nabla F(x,y); \text{ hence } |\delta F(x,y) \leq |\nabla F(x,y). \]

**Proof** We firstly show that, for each \( u \in X_x(1) \), one has \( \Delta_x \sup_{\mathcal{P}_0(R)} F(x,y)(u) \leq \nabla F(x,y). \) The case of \( \Delta F(x,y)(u) = 0 \) is clear; so, it remains to discuss the alternative \( \Delta F(x,y)(u) \neq 0 \). Each \( v \in \Delta F(x,y)(u) \) has the representation (e06); where \( \{\lambda_n \} \subseteq [0,1], \lambda_n \to 0 \) and \( \{w_n\} \subseteq X, w_n \to u \). Denote \( \alpha_n = ||w_n||; n \geq 0 \) [hence \( \alpha_n \to 1 \)] and \( \{z_n = w_n/\alpha_n; n \geq 0, \mu_n = \lambda_n\alpha_n; n \geq 0 \} \) [hence \( \mu_n \to 0 \)]. By the precise definition, we have \( v \in \text{Lim}_{n}(\alpha_n/\mu_n)(y - F(x + \mu_n z_n)) \); and this, along with \( \alpha_n \to 1 \), gives \( v \in \text{Lim}_{n}(1/\mu_n)(y - F(x + \mu_n z_n)) \), where \( \{\mu_n\} \subseteq [0,1], \mu_n \to 0, \{z_n\} \subseteq X_x(1), z_n \to u \). Let \( \varepsilon > 0 \) be arbitrary fixed. From the very choice of \( \{\mu_n\} \), there must be some \( n(\varepsilon) \) with \( n \geq n(\varepsilon) \implies \mu_n < \varepsilon \). Combining with the properties of \( \{z_n\} \) gives
\[ (\forall n \geq n(\varepsilon) : \frac{1}{\mu_n}[y - F(x + \mu_n z_n)] \leq \{ y - F(w) \frac{d(x,w)}{d(x,w)} \}; 0 < d(x,w) < \varepsilon). \]

Denote by \( A(\varepsilon) \) the supremum in the right hand side. We have (from the above) \((1/\mu_n)[y - F(x + \mu_n z_n)] \leq A(\varepsilon), \) for all \( n \geq n(\varepsilon) \); wherefrom (passing to limit as \( n \to \infty \)) \( v \leq A(\varepsilon). \)
As \( v \) is arbitrarily fixed in \( \Delta F(x, y)(u) \), one gets \( \Delta F(x, y)(u) \leq A(\varepsilon) \); hence (by definition) \( \Delta_{\infty} F(x, y)(u) \leq A(\varepsilon) \). Passing to supremum over \( u \in X_{(1)} \), we derive \( \delta F(x, y) \leq A(\varepsilon) \); and from this, the expected relation follows by taking the infimum upon \( \varepsilon > 0 \). □

We may now give the announced differential version of the results in Section 4. Let the function \( \Gamma : X \to R_{+} \) be as in (d01)+(d02); and the multivalued map \( F : X \to \mathcal{P}_{0}(R) \) (from \( X \) to \( R \)) be as in (d03). The differential type "hybrid" condition below is to be considered:

\[
(e09) \text{ each sequence } (x_{n}) \text{ in } \Gamma^{-1}(R_{+}^{0}) \text{ for which } (f(x_{n})) \text{ converges and } |\delta F(x_{n}, f(x_{n}))| \to 0 \text{ has a subsequence } (u_{n}) \text{ with } (\Gamma(u_{n})) \text{ bounded.}
\]

(Here, \( f = \inf[F] \) is the inf-selection of \( F \)). This will be referred to as the differential Palais-Smale condition upon \( F \) with respect to \((\Gamma, f)\). Note that, by Lemma 4 above, (e09) implies (d09). This, along with Theorem 5, gives the following differential coercivity criterion:

**Theorem 6** Assume (in addition) that \( F \) fulfills a differential Palais-Smale condition with respect to \((\Gamma, f)\). Then, \( F \) is necessarily \( \Gamma \)-coercive.

(D) An interesting particular version of the constructions above may be given along the following lines. Let \((X, ||.||), (Y, ||.||)\) be a couple of normed spaces; and \( F : X \to \mathcal{P}_{0}(Y) \) be a multivalued map from \( X \) to \( Y \). For each \((x, y) \in F\), let us introduce a relation \( \Theta F(x, y) \) on \( X \times Y \) as: for \( u \in X \), we say that \( v \in \Theta F(x, y)(u) \) if and only if

\[
(e10) \exists (\lambda_{n}) \subseteq R_{+}^{0}, \lambda_{n} \to 0, \exists (y_{n}) \subseteq Y: \quad (y_{n} \in F(x + \lambda_{n} u); n \geq 0) \text{ and } v = \lim_{n}(1/\lambda_{n})(y - y_{n}).
\]

This also writes, for simplicity

\[
(e11) \exists (\lambda_{n}) \subseteq R_{+}^{0}, \lambda_{n} \to 0: w \in \text{Lim sup}_{\lambda \to 0+}(1/\lambda)(y - F(x + \lambda u)).
\]

By definition, \( \Theta F(x, y) \) will be referred to as the local Dini derivative for \( (x, y) \) at \( u \). As before, the multivalued map \( u \mapsto \Theta F(x, y)(u) \) exists at least in \( u = 0 \) and is positively homogeneous over its domain; cf (5.2)+(5.3). Moreover, we have (by these very conventions)

\[
\Theta F(x, y) \subseteq \Delta F(x, y), \text{ for all } (x, y) \in F. \quad (5.5)
\]

The inclusion may be strict, as simple examples show.

In particular, let \( F : X \to \mathcal{P}_{0}(R) \) be a multivalued map from \( X \) to \( R \). Define, for \((x, y) \in F\)

\[
(e12) \Theta_{\infty} F(x, y)(u) = \sup[\Theta F(x, y)(u)], \quad u \in X.
\]

The obtained map \( u \mapsto \Theta_{\infty} F(x, y)(u) \) is well defined over all of \( X \), with values in \( \tilde{R} \). In addition, we have (by (5.5)) at each \((x, y) \in F\)

\[
\Theta_{\infty} F(x, y)(u) \leq \Delta_{\infty} F(x, y)(u), \text{ for all } u \in X. \quad (5.6)
\]

Finally, let us define for each \((x, y) \in F\)
(e13) $|\theta| F(x, y) = \max\{0, \theta F(x, y) := \sup\{\Theta F(x, y)(u); u \in X(1)\}\}$.

This will be referred to as the local differential strong $d$-slope of $F$ at $(x, y)$. By (5.6) above, it is clear that, for each $(x, y) \in F$,

$$\theta F(x, y) \leq \delta F(x, y); \text{ hence } |\theta| F(x, y) \leq |\delta| F(x, y).$$

Having these precise, let the function $\Gamma : X \to R_+$ be as in (d01)+(d02) and the multivalued map $F : X \to P_0(R)$ (from $X$ to $R$) be as in (d03). The differential type "hybrid" condition below is to be considered:

(e14) each sequence $(x_n)$ in $\Gamma^{-1}(R_1)$ for which $(f(x_n))$ converges and

$$|\theta| F(x_n, f(x_n)) \to 0 \text{ has a subsequence } (u_n) \text{ with } (\Gamma(u_n)) \text{ bounded.}$$

This will be referred to as the local differential Palais-Smale condition upon $F$ with respect to $(\Gamma, f)$. Note that by (5.7) above, (e14) implies (e09). Combining with Theorem 6 gives the following differential coercivity criterion:

**Theorem 7** Assume (in addition) that $F$ fulfills a local differential Palais-Smale condition with respect to $(\Gamma, f)$. Then, $F$ is necessarily $\Gamma$-coercive.

(E) The obtained result is for the moment a particular version of the preceding one (Theorem 6). So, it is legitimate to ask whether this (logical) inclusion may be reversed. A concrete circumstance yielding such a relation it to be described as

(e15) $F$ is locally Lipschitz: $\forall x \in X, \exists \varepsilon(x) > 0, \exists \nu(x) > 0;

F(x') \subseteq F(x^0) + \nu(x)||x' - x^0||[-1, 1], \forall x', x^0 \in X(x, \varepsilon(x)).$

The following auxiliary fact will clarify this.

**Lemma 5** Let the multivalued map $F : X \to P_0(R)$ (from $X$ to $R$) be locally Lipschitz. Then, for each $(x, y) \in F$,

$$\Delta F(x, y) \subseteq \Theta F(x, y); \text{ hence } \Delta F(x, y) = \Theta F(x, y)$$

$$\Delta F(x, y) = \Theta F(x, y), \delta F(x, y) = \theta F(x, y), \quad |\delta| F(x, y) = |\theta| F(x, y).$$

**Proof** Let $(u, v) \in \Delta F(x, y)$ be arbitrary fixed; hence $v \in \Delta F(x, y)(u)$ has the representation (e05). Take some rank $p = p[x]$ according to

(e16) $\lambda_n||w_n|| \leq \varepsilon(x), \lambda_n||u|| \leq \varepsilon(x), \text{ for all } n \geq p.$

By (e15), the sequence $(y_n; n \geq 0)$ appearing there admits the decomposition (for all $n \geq p$):

$y_n = y^0_n + \nu(x)\lambda_n||w_n - u||z_n; \text{ where } y^0_n \in F(x + \lambda_n u), z_n \in [-1, 1], \text{ for all such } n.$

So, $v = \lim_{n}(1/\lambda_n)(y - y^0_n) - \lim_{n}(\nu(x)||w_n - u||z_n) = \lim_{n}(1/\lambda_n)(y - y^0_n);$ which tells us that $v \in \Theta F(x, y)(u)$; or, equivalently: $(u, v) \in \Theta F(x, y).$ This proves (5.8); hence, (5.9) as well. □

In other words, (e09) and (e14) are identical under (e15). On the other hand, (e15) yields the upper semicontinuous condition (e08); which in turn gives (as precise in Section 3) the closed (graph) condition (e05) whenever $F$ has nonempty closed values (cf. (e07)). This, along with (e02)+(e05) $\implies$ (e03) (cf. Section 3) yields the following coercivity criterion:
Theorem 8 Let the function \( \Gamma : X \rightarrow \mathbb{R}^+ \) be as in (d01)+(d02); and the multivalued map \( F : X \rightarrow \mathcal{P}_0(\mathbb{R}) \) (from \( X \) to \( \mathbb{R} \)) be such that (c02)+(c07)+(e15) hold, as well as (e09) (or equivalently, (e14)). Then, \( F \) is \( \Gamma \)-coercive.

In particular, when (c07) is taken in the stronger sense

(e17) \( F \) is (nonempty) compact valued: \( (\emptyset \neq F(x))=\text{compact}, \forall x \in X \)

Theorem 8 is just the main result in Kristaly and Varga [17]. Some “conical” extensions of these results are directly obtainable under the lines in Motreanu and Turinici [21]; further aspects will be delineated elsewhere.

References


