Almost-periodicity of solutions of the abstract wave equation

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Introduction

In the present paper we shall give an abstract version (in terms of operator differential equations) of the well-known result about almost-periodicity in $H^1(\Omega) \times L^2(\Omega)$ norm ($\Omega$ being a bounded open set in $\mathbb{R}^n$) of the vector-function $\{u(x,t), u_t(x,t)\}, x \in \Omega, t \in \mathbb{R}$, where $u_t = \sum_{j=1}^n \frac{\partial}{\partial x_j} \left( a_{ij}(x) \frac{\partial u}{\partial x_i} \right)$ (the wave-equation) and $u$ satisfies homogeneous boundary conditions on $\partial \Omega$ (see [1], [2]). We therefore consider a Hilbert space $H$ and then a linear self-adjoint operator $L$ with dense domain $D(L) \subset H$ and with range in $H$, such that

$$\langle Lh, Lh \rangle \geq \gamma \|h\|^2_H \quad (\text{some } \gamma > 0) \text{ holds true }, \forall h \in D(L).$$

(0.1)

The abstract wave equation is the second order equation for $H$-valued functions $u(\cdot)$:

$$u''(t) = -L^2u(t), t \in \mathbb{R}$$

(0.2)

We will be concerned here with a class of weak solutions of (0.2); they are defined as functions $u(\cdot) : \mathbb{R} \to D(L), u(\cdot) \in C^1(\mathbb{R}; H), Lu \in C(\mathbb{R}; H)$, such that the integral relation

$$\int_{\mathbb{R}} \langle u'(t), h(t) \rangle_H dt = \int_{\mathbb{R}} \langle Lu(t), Lh(t) \rangle_H dt$$

(0.3)

holds, $\forall h \in C^1(\mathbb{R}; H), Lh \in C_0(\mathbb{R}; H)(C^1(\mathbb{R}; H)$ and $C_0(\mathbb{R}; H)$ are functions with compact support in $\mathbb{R}$).
Supplementary assumptions are made on the operator \( L \) as follows: there exist sequences \( \{\varepsilon_n\} \subset D(L) \) and \( \{\lambda_n\} \subset \mathbb{R}^+ \), such that:

\[
0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots \rightarrow +\infty, \quad (\lambda_n \varepsilon_n, \varepsilon_n) = \delta_{2n}, (L\varepsilon_n, L\varepsilon_n) = \lambda_n^2 \varepsilon_n
\]

\[
= \lambda_n^2 (e_n, \lambda_n, \forall \varepsilon_n \in D(L); \|L\varepsilon_n - \sum_{j=1}^{n} (\varepsilon_j, e_j) L e_j\| \rightarrow 0 \text{ as } n \rightarrow \infty, \forall \varepsilon_n \in D(L) \quad (0.4)
\]

Our main result which is established in this chapter says that under the above assumptions \((0.1) - (0.3) - (0.4)\) the function \( L u(\cdot) \) is (Bochner) almost-periodic, \( \mathbb{R} \rightarrow H \) while the function \( u(\cdot) \) is almost-periodic, \( \mathbb{R} \rightarrow H \).

(this is the result stated as "Teorema 1" in [3]; the proof now given is different from that indicated in [3] - p. 13 and similar to the proof which appears in Amerio-Prouse [1]).

1.

We pass now to give the proof of the above stated results; it consists in several steps. First we have

**PROPOSITION 1.** For any \( \varepsilon \in D(L) \) we have that

\[
\|\varepsilon \varepsilon - \sum_{j=1}^{N} (\varepsilon_j, e_j) e_j\| \rightarrow 0 \quad \text{as} \quad N \rightarrow \infty, \quad \|\varepsilon\| = \sum_{j=1}^{\infty} \|\varepsilon_j\|^2 \quad (1.1)
\]

In fact, using (1.1) we get

\[
\gamma \|\varepsilon \varepsilon - \sum_{j=1}^{N} (\varepsilon_j, e_j) e_j\| \leq \|\varepsilon \varepsilon - \sum_{j=1}^{N} (\varepsilon_j, e_j) L e_j\|
\]

if we use (0.4) we get the first assertion. Next, if we denote \( h_N = \sum_{j=1}^{N} (\varepsilon_j, e_j) e_j \) we see that \( ||h_N||^2 = (h_N, h_N) = \sum_{j=1}^{N} \|\varepsilon_j\|^2 \) as \( h_N \rightarrow h \) for \( N \rightarrow \infty \), it follows that \( ||h_N||^2 \rightarrow ||h||^2 \), hence \( \sum_{j=1}^{\infty} \|\varepsilon_j\|^2 = ||h||^2 \). Next, we prove

**PROPOSITION 2.** For any \( \varepsilon \in D(L) \) the equality

\[
\|L\varepsilon\|^2 = \sum_{j=1}^{\infty} \lambda_j^2 (\varepsilon_j, e_j)^2 \quad (1.2)
\]

holds true.
In fact, if \( h_N = \sum_{j=1}^{N}(h,e_j)e_j \) and \( g_N = h - h_N \), we obtain
\[
||Lg_N||^2 = (Lh - Lh_N, Lh - Lh_N) = ||Lh||^2 - \left( Lh, \sum_{j=1}^{N}(h,e_j)Le_j \right)
- \sum_{j=1}^{N}(h,e_j)Le_j, Lh \right) + \sum_{j=1}^{N}(h,e_j)Le_j, Lh \right)
= ||Lh||^2 - \sum_{j=1}^{N}(h,e_j)(Lh,Le_j) - \sum_{j=1}^{N}(h,e_j)(Le_j,Lh) + \sum_{j=1}^{N}||e_j||^2 \lambda_j^2
= ||Lh||^2 - \sum_{j=1}^{N}(h,e_j)(Lh,Lh) - \sum_{j=1}^{N}(h,e_j)(Le_j,Lh) + \sum_{j=1}^{N}||e_j||^2 \lambda_j^2
= ||Lh||^2 - \sum_{j=1}^{N}||e_j||^2 \lambda_j^2
\]

Also we see that \( Lg_N = Lh - Lh_N = Lh - \sum_{j=1}^{N}(h,e_j)Le_j \) and from (0.4) we derive:
\( ||Lg_N||^2 \to 0 \) as \( N \to \infty \) which means, in view of the above computation, that
\( \sum_{j=1}^{\infty}||e_j||^2 \lambda_j^2 = ||Lh||^2. \)

We can now consider the demonstration of the main result. Assume therefore the relation (0.3) and then take test-functions \( h(t) \) of the special form: \( h(t) = \zeta(t)e_n \), where \( \zeta(t) \in C_0^0(\mathbb{R}). \) We get accordingly
\[
\int_{\mathbb{R}}(u'(t),e_n\zeta'(t))dt = \int_{\mathbb{R}}(Lu(t),Le_n\zeta(t))dt, n = 1, 2, \ldots \tag{1.3}
\]

With the notation \( (u(t),e_n) = u_n(t) \) and using again (0.4) we obtain
\[
\int_{\mathbb{R}}u_n'(t)\zeta'(t)dt = \int_{\mathbb{R}}\lambda_n^2u_n(t)\zeta(t)dt, \forall n \in \mathbb{N}, \forall \zeta(\cdot) \in C_0^0(\mathbb{R})
\]

From the continuity of the function \( \lambda_n^2u_n(t) \) we derive: (elementary "distribution theory"): \( u_n' \) exists and \( u_n'(t) = -\lambda_n^2u_n(t), \forall t \in \mathbb{R}, n \in \mathbb{N}. \) Therefore we get
\[
u_n(t) = (\cos \lambda_n t)u_n(0) + \frac{1}{\lambda_n}(\sin \lambda_n t)u_n'(0) \tag{1.4}
\]

Next, from Proposition 1, we have that:
\[
u(t) = \sum_{j=1}^{\infty}u_j(t)e_j \quad \text{in } H \text{- norm, for any } t \in \mathbb{R}. \tag{1.5}
\]

Actually, as we see below, the series (1.5) is \( H \)-uniformly convergent on \( \mathbb{R} \); the same holds for the series \( \sum_{j=1}^{\infty}u_j(t)Le_j \) as well for \( \sum_{j=1}^{\infty}u_j'(t)e_j. \)
From (1.4) we obtain: \( |u_n(t)| \leq |u_n(0)| + \frac{1}{\lambda_n} |u_n'(0)| \) and
\[
|u_n(t)|^2 \leq 2|u_n(0)|^2 + \frac{1}{\lambda_n^2} |u_n'(0)|^2, \quad \forall t \in \mathbb{R} \tag{1.6}
\]
The series \( \sum_{n=1}^{\infty} |u_n(0)|^2 = \sum_{n=1}^{\infty} (|u(0), e_n|^2) \) is convergent to \( ||u(0)||^2 \) (Proposition 1).

Also, using (0.1) and (0.4) we find that \( \gamma \|e_n\|^2 \leq (Le_n, Le_n) = \lambda_n^2 \), hence \( \lambda_n^2 \geq \gamma, n = 1, 2, \ldots \).

Therefore the series \( \sum_{n=1}^{\infty} \frac{1}{\lambda_n^2} |u_n'(0)|^2 \) is \( \leq \sum_{n=1}^{\infty} \frac{1}{\lambda_n^2} (|u'(0)|, e_n)^2 = \frac{1}{\gamma} ||u'(0)||^2 \), hence it is also convergent. This gives uniform convergence on \( \mathbb{R} \) of the series \( \sum_{n=1}^{\infty} |u_j(t)|^2 \).

Next, we estimate the expression
\[
\left\| \sum_{n=N+1}^{N+p} u_j(t)Le_j \right\|^2
\]
which is also (in view of (0.4)),
\[
\left( \sum_{n=N+1}^{N+p} u_j(t)Le_j, \sum_{n=N+1}^{N+p} u_j(t)Le_j \right) = \sum_{n=N}^{N+p} |u_j(t)|^2 \lambda_j^2 \tag{1.7}
\]
From (1.6) it follows that
\[
\lambda_j^2 |u_j(t)|^2 \leq 2(\lambda_j^2 |u_j(0)|^2 + |u_j'(0)|^2).
\]
Then we use Proposition 2, and we get
\[
||Lu(0)||^2 = \sum_{j=1}^{\infty} \lambda_j^2 ((u(0), e_j)^2 = \sum_{j=1}^{\infty} \lambda_j^2 |u_j(0)|^2, \tag{1.8}
\]
again a convergent series.

This way we have established uniform convergence on \( \mathbb{R} \) of the series \( \sum_{j=1}^{\infty} u_j(t)Le_j \).

Finally, \( u_j'(t) = -\frac{1}{\lambda_j} \sin(\lambda_j t)u_j(0) + \cos(\lambda_j t)u_j'(0) \) and accordingly \( |u_j'(t)| \leq 1 |u_j(0)| + 1 |u_j'(0)| \) and \( |u_j'(t)|^2 \leq 2(\lambda_j^2 |u_j(0)|^2 + |u_j'(0)|^2) \) and we use same arguments to establish uniform convergence on \( \mathbb{R} \) of the series \( \sum_{j=1}^{\infty} u_j'(t)e_j \).

It remains only to see that
\[
Lu(t) = \sum_{j=1}^{\infty} u_j(t)Le_j \quad \text{and} \quad u'(t) = \sum_{j=1}^{\infty} u_j'(t)e_j \tag{1.9}
\]
We know that \( u(t) = \sum_{j=1}^{\infty} u_j(t)e_j \) and that \( L \) is closed.
From the convergence of $\sum_{\infty} u_j(t)Le_j$ we derive that $Lu(t) = \sum_{\infty} u_j(t)Le_j$; hence $Lu$ is almost-periodic, $\mathbb{R} \rightarrow H$.

Finally we know that both series: $\sum_{\infty} u_j(t)e_j$ and $\sum_{\infty} u'_j(t)e_j$ are uniformly convergent. We get that $u'(t) = \sum_{\infty} u'_j(t)e_j$, and is again an almost-periodic function.

References


Note. This article appears in detailed form for the first time. It gives complete proof to a statement in [3] (Teorema 1).