ON A SINGULAR PERTURBATION PROBLEM
WITH MIXED CONDITION

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INTRODUCTION

In the square \( \Omega = [0, \pi] \times [0, \pi] \), we consider the following Dirichlet problem for the second order elliptic equations:

\[
-\varepsilon^2 \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} + u = f_\varepsilon \quad \text{in} \quad \Omega,
\]

\( (P_\varepsilon) \)

\[
u(0,y) = u(\pi,y) = 0 \quad \text{for} \quad 0 < y < \pi,
\]

\[
\frac{\partial}{\partial y} u(x,0) = \frac{\partial}{\partial y} u(x,\pi) = 0 \quad \text{for} \quad 0 < x < \pi,
\]

where \( f_\varepsilon \) is in the space \( L^2(\Omega) \).

We begin this paper by proving that as \( \varepsilon \) converges to \( f \) in \( L^2(\Omega) \), the solution of the problem \( (P_\varepsilon) \) converges in \( L^2(\Omega) \) to the solution of the following problem:

\[
-\frac{\partial^2 u}{\partial y^2} + u = f \quad \text{in} \quad \Omega,
\]

\( (P_0) \)

\[
\frac{\partial}{\partial y} u(x,0) = \frac{\partial}{\partial y} u(x,\pi) = 0 \quad \text{for} \quad 0 < x < \pi.
\]

On the other hand, if \( f_\varepsilon \approx f \) is sufficiently smooth, we construct the asymptotic expansion of the solution \( u_\varepsilon \) of the problem \( (P_\varepsilon) \).

1. Convergence of the solution of the problem \( (P_\varepsilon) \), if \( f_\varepsilon \) converges in \( L^2(\Omega) \)

In the square \( \Omega = [0, \pi] \times [0, \pi] \), we consider the following Dirichlet problem for the second order elliptic equations:

\[
-\varepsilon^2 \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} + u = f_\varepsilon \quad \text{in} \quad \Omega,
\]

\( (P_\varepsilon) \)

\[
u(0,y) = u(\pi,y) = 0 \quad \text{for} \quad 0 < y < \pi,
\]

\[
\frac{\partial}{\partial y} u(x,0) = \frac{\partial}{\partial y} u(x,\pi) = 0 \quad \text{for} \quad 0 < x < \pi.
\]

Proposition 1:

If \( u_\varepsilon \) is the solution of problem \( (P_\varepsilon) \) and if \( f_\varepsilon \) converges to \( f \) in \( L^2(\Omega) \) then \( u_\varepsilon \) converges in \( L^2(\Omega) \) to the solution of the boundary problem \( (P_0) \).
\[-\frac{\partial^2}{\partial y^2}u + u = f \quad \text{in } \Omega,\]

(P0) \[\frac{\partial}{\partial y}u(x,0) = \frac{\partial}{\partial y}u(x,\pi) = 0 \quad \text{for } 0 < x < \pi.\]

Proof:

Let's take \((\Psi_{n,m})\) an orthonormal basis of \(L^2(\Omega)\) (see[8]):

(1.1) \[\Psi_{n,m}(x,y) = (2/\pi)\sin(nx)\cos(my) \quad \text{and} \quad \Psi_{n,0}(x,y) = \frac{\sqrt{2}}{\pi}\sin(nx).\]

Taking the scalar product of the two sides of \((P_e)\) and \((P_0)\) by \(\Psi_{n,m}\) and using the fact that \(\frac{\partial}{\partial y}u(x,0) = \frac{\partial}{\partial y}u(x,\pi) = u_x(0,y) = u_x(\pi,y) = 0\) and \(\frac{\partial}{\partial y}u(x,\pi) = \frac{\partial}{\partial y}u(x,\pi) = 0\) one has

(1.2) \[(e^2n^2 + m^2 + 1)u_{n,m,\varepsilon} = f_{n,m,\varepsilon},\]

(1.3) \[(m^2 + 1)u_{n,m} = f_{n,m},\]

with

(1.4) \[u_{n,m,\varepsilon} = \int_{\Omega} u(x,y)\Psi_{n,m}(x,y) dxdy, \quad f_{n,m,\varepsilon} = \int_{\Omega} f(x,y)\Psi_{n,m}(x,y) dxdy\]

and

\[u_{n,m} = \int_{\Omega} u(x,y)\Psi_{n,m}(x,y) dxdy, \quad f_{n,m} = \int_{\Omega} f(x,y)\Psi_{n,m}(x,y) dxdy.\]

Consequently, the solution of problems \((P_e)\) and \((P_0)\) are given respectively by:

(1.5) \[u(x,y) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \left(\frac{f_{n,m,\varepsilon}}{\mu_{n,m,\varepsilon}}\right) \Psi_{n,m}(x,y),\]

and

(1.6) \[u(x,y) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \left(\frac{f_{n,m}}{\mu_{n,m}}\right) \Psi_{n,m}(x,y),\]

where:

(1.7) \[\mu_{n,m,\varepsilon} = e^2n^2 + m^2 + 1.\]

We remark that for every \((n,m)\) in \(\mathbb{N}^* \times \mathbb{N}^*\):

(1.8) \[\lim_{\varepsilon \to 0} u_{n,m,\varepsilon} = \lim_{\varepsilon \to 0} \left(\frac{f_{n,m,\varepsilon}}{\mu_{n,m,\varepsilon}}\right) = f_{n,m},\]

This prove the weak convergence of \(u_{n,\varepsilon}\) to \(u\) in \(L^2(\Omega)\).

On the other hand, we have:

(1.9) \[|u_{n,m,\varepsilon}| = \left|\frac{f_{n,m,\varepsilon}}{\mu_{n,m,\varepsilon}}\right| \leq C \left|\frac{f_{n,m}}{\mu_{n,m}}\right|,
\]

where \(C\) is a constant independent of \(\varepsilon\) and \((n,m)\) (this can be easily seen from the convergence of \(\mu_{n,m,\varepsilon}\) to \(\mu_{n,m}\) in \(L^2(\Omega)\)).

So we are able to apply the Lebesgue dominated convergence theorem:
\[
(1.10) \quad \lim_{\epsilon \to 0} \| u_\epsilon \|^2 = \lim_{\epsilon \to 0} \sum_{n \geq 1} \sum_{m \geq 1} | \frac{f_{n,m}}{\vartheta_{n,m,\epsilon}} |^2 = \sum_{n \geq 1} \sum_{m \geq 0} | \frac{f_{n,m}}{\vartheta_{n,m,0}} |^2 = \| u \|^2,
\]
which will complete the proof of the proposition 1([2]).

**Remark 1:**

If \( f_\epsilon \) converges to \( f \) in \( H^1(\Omega) \), without having \( f(0,y) = f(\pi,y) = 0 \), then the \( (u_\epsilon) \) solution doesn't converge in \( H^1(\Omega) \). Indeed, in the \((P_\epsilon)\) and \((P_0)\) problems, let's take
\[
(1.11) \quad f_\epsilon(x,y) = f(x,y) = g(x) \lambda(y) \in H^1(\Omega) \text{ and } g \notin H^1(\partial\Omega, \pi).
\]
Then the solution \( (u_\epsilon) \) of \((P_\epsilon)\) converges in \( L^2(\Omega) \) to the solution of \((P_0)\), with
\[
(1.12) \quad u(x,y) = g(x)v(y),
\]
where \( v \) is the solution of the following boundary problem:
\[
(\Omega) \quad -\frac{\partial^2}{\partial y^2} u + u = h \quad \text{for } 0 < y < \pi,
\]
\[
(\partial \Omega) \quad \frac{\partial}{\partial y} u(0) = \frac{\partial}{\partial y} u(\pi) = 0.
\]
As \( g \notin H^1(\partial\Omega, \pi) \) implies that \( u \notin E \), where
\[
(1.13) \quad E = \{ u \in H^1(\Omega) ; u(0,y) = u(\pi,y) = 0 \}. \]
Consequently, if \( u_\epsilon \) converges in \( H^1(\Omega) \), \( E \) being a closed subset of \( H^1(\Omega) \), \( u \) must be in \( E \), which isn't the case.

2. **Convergence of the solution of the problem \((P_0)\), if \( f_\epsilon = f \) is smooth**

We consider again the boundary problems:
\[
(\Omega) \quad -\frac{\partial^2}{\partial y^2} u_\epsilon + u_\epsilon = f \quad \text{in } \Omega,
\]
\[
(P_0) \quad u_\epsilon(0,y) = u_\epsilon(\pi,y) = 0 \quad \text{for } 0 < y < \pi,
\]
\[
\frac{\partial}{\partial y} u_\epsilon(x,0) = \frac{\partial}{\partial y} u_\epsilon(x,\pi) = 0 \quad \text{for } 0 < x < \pi,
\]
and
\[
(\Omega) \quad -\frac{\partial^2}{\partial y^2} u + u = f \quad \text{in } \Omega,
\]
\[
(P_0) \quad \frac{\partial}{\partial y} u(x,0) = \frac{\partial}{\partial y} u(x,\pi) = 0 \quad \text{for } 0 < x < \pi,
\]
where \( f \) belongs to \( L^2(\Omega) \) and verifies
(H): For every \( y \in [0, \pi] \), the function \( f(\cdot, y) \in C^\infty([0, \pi]) \).

Proposition 2

If \( f \) verifies the hypothesis (H), and if \( u_{\epsilon} \) and \( u \) are the respective solutions of \((P_{\epsilon})_1\) and \((P_0)_1\) problems, then

\[
\| u_{\epsilon} - u \| \leq C \sqrt{\epsilon} ,
\]

where \( \| \cdot \| \) is the \( L^2(\Omega) \) norm and \( C \) is a constant.

Proof:

Following the first section results, \( u_{\epsilon} \) converges to \( u \) in \( L^2(\Omega) \), so there is a boundary conditions loose near \([0] \times ]0, \pi[\) and \([\pi] \times ]0, \pi[\), and consequently \( u_{\epsilon} \) can be written as

\[
u_{\epsilon}(x, y) = v_{\epsilon}(x, y) + \epsilon^2 v_{\epsilon}(x, y) + \epsilon^4 v_{\epsilon}(x, y) + \cdots
\]

where \( \omega_{\epsilon} \) and \( \chi_{\epsilon} \) are the limits layers phenomena near \([0] \times ]0, \pi[\) and \([\pi] \times ]0, \pi[\), respectively.

To determine the expansion of \( v_{\epsilon} \), we use the recurrent relation exposed by A. Nayfeh [6]:

\[
u_0(x, y) = v_0(x, y) + \epsilon^2 v_0(x, y) + \epsilon^4 v_0(x, y) + \cdots
\]

Reporting the expansion in \((P_{\epsilon})_1\) and gathering the terms in \( \epsilon \), we obtain:

\[
\left( -\frac{\partial^2}{\partial y^2} + v_0 \right) + \sum_{k \geq 1} \epsilon^k \left( -\frac{\partial^2}{\partial y^2} v_k + \frac{\partial^2}{\partial x^2} v_{k-1} \right) = f
\]

Consequently

\[
-\frac{\partial^2}{\partial y^2} v_0 + v_0 = f \quad \text{in } \Omega ,
\]

\[
-\frac{\partial^2}{\partial y^2} v_k + v_k = \frac{\partial^2}{\partial x^2} v_{k-1} \quad \text{for } k \geq 1
\]

To \( v_k \) we impose the conditions

\[
\frac{\partial}{\partial y} v_k(x, 0) = \frac{\partial}{\partial y} v_k(x, \pi) = 0 \quad \text{for } k \in \mathbb{N}, \ y \in [0, \pi].
\]

So, \( v_0 \) is the solution of the \((P_0)_1\) problem (i.e. \( v_0 = u \)).

The problems (2.6) and (2.7) admit unique solution following the fact that \( f \) verifies the hypothesis (H). On the other hand, we have for \( \omega_{\epsilon} \):

\[
\omega_\epsilon(x, y) = \omega_0(x, y, \epsilon) + \epsilon^2 \omega_1(x, y, \epsilon) + \epsilon^4 \omega_2(x, y, \epsilon) + \cdots
\]

Here \( \omega_{\epsilon} \) doesn't occur near \([0] \times ]0, \pi[\), and so, it is natural to look to \((\omega_0)\) as follows
(2.9) \[ \omega_k(x, y, z) = \omega_k(t, y) \text{ with } t = e^{-1}z. \]

This variable change induces a change in the \( A_k \) operator:

(2.10) \[ A_k = -\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial y^2} + I_d, \]

where \( I_d \) is the identity operator.

Let's apply the operator \( A_k \) to the \( \omega_k \) expansion:

(2.11) \[ A_k[\omega_0 + \epsilon^0 \omega_1 + \epsilon^4 \omega_2 + \cdots] = 0, \]

which implies

(2.12) \[ -\frac{\partial^2}{\partial t^2} \omega_k + \frac{\partial^2}{\partial y^2} \omega_k + \omega_k = 0 \text{ in } \Omega \text{ for } k \in \mathbb{N}, \]

with the following boundary conditions:

(2.13) \[ \frac{\partial}{\partial y} \omega_k(t, 0) = \frac{\partial}{\partial y} \omega_k(t, \pi) = 0 \text{ for } t \geq 0, \]

\[ \omega_k(0, y) + \omega_k(\pi, y) = 0 \text{ for } 0 < y < \pi, \]

\[ \omega_k(t, y) = 0 \text{ when } t \to \infty. \]

Similarly, we have for \( x_k \):

(2.14) \[ x_k(x, y) = x_0(x, y, c) + \epsilon^0 x_1(x, y, c) + \epsilon^4 x_2(x, y, c) + \cdots, \]

and operating the following variable change

(2.15) \[ x_k(x, y, c) = x_k(x, y, s = c^{-1}(\pi - x)), \]

the operator \( A_k \) will be

(2.16) \[ A_k = -\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial y^2} + I_d. \]

So, we find that

(2.17) \[ -\frac{\partial^2}{\partial t^2} x_k - \frac{\partial^2}{\partial y^2} x_k + x_k = 0 \text{ in } \Omega \text{ for } k \in \mathbb{N}, \]

with

(2.18) \[ \frac{\partial}{\partial y} x_k(t, 0) = \frac{\partial}{\partial y} x_k(t, \pi) = 0 \text{ for } t \geq 0, \]

\[ x_k(0, y) + x_k(\pi, y) = 0 \text{ for } 0 < y < \pi. \]

For the case \( k = 0 \), the solutions of the problems (2.12), (2.13) and (2.17),

(2.18) are respectively:

(2.19) \[ \omega_0(t, y) = -\sum_{p \geq 0} \left( \frac{f_p(0)}{1 + p}\right) \phi_p(y) \exp(-t \sqrt{1 + p^2}) \]

(2.20) \[ x_0(x, y) = -\sum_{p \geq 0} \left( \frac{f_p(x)}{1 + p}\right) \phi_p(y) \exp(-x \sqrt{1 + p^2}). \]
where
\[
\phi_p(y) = \sqrt{2} \cos(py); \quad \phi_d(y) = \sqrt{2} \quad \text{and} \quad f_p(x) = \int_0^x f(x,y) \phi_p(y) \, dy.
\]

Consequently,
\[
(2.21) \quad u_k = (u + x_0 + \omega_k) + \sum_{k \geq 1} \epsilon^{2k} (u_k + x_k + \omega_k),
\]
which will complete the proof of the inequality (2.1).

**Remark 2**

If we consider the problem \((P_{x1})\), with \(\beta \) verifying the hypothesis \((H)\), while
\[
f(0,y) = f(\pi,y) = 0, \quad \text{for every } p \text{ in } N, f_p(0) = f_p(\pi) = 0
\]
and consequently the solutions of problems (2.12), (2.13) and (2.17), (2.18) for \(k = 0\) are identically zero, then the inequality (2.1) takes the form
\[
(2.22) \quad \|u_k - u\| \leq C\epsilon^2.
\]

**References:**

2. T. Benkirane, Étude d'un problème de perturbation singulière en une dimension, à paraître.