NOTES ON TOPOLOGICAL APPLICATIONS
OF REGULAR OR $\mathcal{C}$-SMOOTH MEASURES TO WALLMAN TYPE SPACES

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1. INTRODUCTION. Let $X$ be an arbitrary set, and $\mathcal{L}$ a lattice of subsets of $X$. It is assumed throughout the paper that $\emptyset, X \in \mathcal{L}$.

We adhere to the customary lattice – topological definitions which can be found for example in [1],[2],[4],[7] and [10]. Here, we just note some of the measure theoretic equivalents. For this purpose we introduce the following notations: $\mathcal{A}(\mathcal{L})$ denotes the algebra generated by $\mathcal{L}$, and $I(\mathcal{L})$ the set of non-trivial zero-one valued finitely additive measures on $\mathcal{A}(\mathcal{L})$. $I_{\mathcal{L}}(\mathcal{L})$ the set of $\mathcal{L}$-regular measures of $I(\mathcal{L})$, where $\mu \in I(\mathcal{L})$ is $\mathcal{L}$-regular if for any $A \in \mathcal{A}(\mathcal{L})$

$$\mu(A) = \sup \{ \mu(L)/L \subseteq A \in \mathcal{L} \},$$

$I_{\mathcal{L}}(\mathcal{L})$ the set of $\mathcal{L}$-smooth measures of $I(\mathcal{L})$ on $\mathcal{L}$, where $\mu \in I(\mathcal{L})$ is $\mathcal{L}$-smooth on $\mathcal{L}$ if for all sequences $\{L_n\}$ of sets of $\mathcal{L}$ with $L_n \uparrow \emptyset$, $\mu(L_n) \to 0$. $I_{\mathcal{L}}(\mathcal{L})$ the set of $\mathcal{L}$-regular measures of $I(\mathcal{L})$. $\mathcal{K}(\mathcal{L}) = \{ \mathcal{T}, \text{defined on } \mathcal{L}, \text{non-trivial,} \text{monotone, and } \mathcal{T}(\mathcal{A} \cap \mathcal{B}) = \mathcal{T}(\mathcal{A}) \mathcal{T}(\mathcal{B}), \mathcal{A}, \mathcal{B} \in \mathcal{L} \}$ the set of all premeasures on $\mathcal{L}$. $\mathcal{K}_{\mathcal{L}}(\mathcal{L})$ is the set of all $\mathcal{L}$-smooth on $\mathcal{L}$.

Note that there exists a one-to-one correspondence between:

$\mathcal{L}$-filters $\mathcal{F}$ and elements of $\mathcal{K}(\mathcal{L})$ given by $\mathcal{T}(\mathcal{L}) = 1$ iff $\mathcal{L} \in \mathcal{F}$.

All elements of $I(\mathcal{L})$ and all prime $\mathcal{L}$-filters, given by:

for any $\mu \in I(\mathcal{L})$ associate the prime $\mathcal{L}$-filter given by:

$$\mathcal{F} = \{ A \subseteq \mathcal{L} : \mu(A) = 1 \}.$$ 

All elements of $I_{\mathcal{L}}(\mathcal{L})$ and all $\mathcal{L}$-ultrafilters, given by the following rule: with each $\mathcal{L}$-ultrafilter $\mathcal{F}$ we associate the zero-one valued measure defined on $\mathcal{A}(\mathcal{L})$ by:

$$\mu(E) = \begin{cases} 1 \text{ if there exists } A \in \mathcal{F}, \forall E \subseteq A \in \mathcal{E} \\ 0 \text{ if there exists } A \in \mathcal{F}, \forall E \subseteq A \notin \mathcal{E} \end{cases}.$$
The support of \( \mu \in I(\mathcal{L}) \) is \( S(\mu) = \bigcap \{ L \in \mathcal{L} : \mu(L) = 1 \} \).

With this notation, we now note: \( \mathcal{L} \) is compact iff \( S(\mu) \neq \emptyset \) for every \( \mu \in I(\mathcal{L}) \). \( \mathcal{L} \) is countably compact iff \( I_R(\mathcal{L}) \subseteq \mathcal{G}_R(\mathcal{L}) \). \( \mathcal{L} \) is normal iff for each \( \mu \in I(\mathcal{L}) \), there exists a unique \( \nu \in I_R(\mathcal{L}) \) such that \( \mu \leq \nu \) (\( \mathcal{L} \)) i.e. \( \mu(L) \leq \nu(L) \) for all \( L \in \mathcal{L} \). \( \mathcal{L} \) is regular iff whenever \( \mu_1, \mu_2 \in I(\mathcal{L}) \) and \( \mu_1 \leq \mu_2 \), then \( S(\mu_1) = S(\mu_2) \). \( \mathcal{L} \) is replete iff for any \( \mu \in I_R(\mathcal{L}) \), \( S(\mu) \neq \emptyset \). \( \mathcal{L} \) is prime-complete iff for any \( \mu \in I_R(\mathcal{L}) \), \( S(\mu) \neq \emptyset \). \( \mathcal{L} \) is Lindelöf iff for any \( \mathcal{T} \in \mathcal{T}_0(\mathcal{L}) \), \( S(\mathcal{T}) \neq \emptyset \).

2. THE SPACES \( I_R(\mathcal{L}) \) AND THE LATTICES \( \mathcal{W}_0(\mathcal{L}) \), \( \mathcal{V}_0(\mathcal{L}) \)

We consider in this section the important space \( I_R(\mathcal{L}) \); for \( A \in \mathcal{A}(\mathcal{L}) \) define \( \mathcal{W}_0(A) = \{ \mu \in I_R(\mathcal{L}) : \mu(A) = 1 \} \). Then, assuming \( \mathcal{L} \) is disjunctive, \( \mathcal{W}_0(\mathcal{L}) = \{ \mathcal{W}_0(L) / L \in \mathcal{L} \} \) is a lattice in \( I_R(\mathcal{L}) \) isomorphic to \( \mathcal{L} \), under the map \( L \to \mathcal{W}_0(L) \), \( L \in \mathcal{L} \), and \( \mathcal{A}(\mathcal{W}_0(\mathcal{L})) = \mathcal{W}_0(\mathcal{A}(\mathcal{L})) \). Also the map \( \mu \to \mu' \), where \( \mu' = \mu'(A) \), \( A \in \mathcal{A}(\mathcal{L}) \) is a bijection between \( I_R(\mathcal{L}) \) and \( I_R(\mathcal{W}_0(\mathcal{L})) \). It is well known that \( \mathcal{W}_0(\mathcal{L}) \) is replete and is a basis for the closed sets \( \mathcal{V}_0(\mathcal{L}) \), all arbitrary intersections of sets of \( \mathcal{W}_0(\mathcal{L}) \). It is this topological space \( I_R(\mathcal{L}), \mathcal{V}_0(\mathcal{L}), \) and lattice \( \mathcal{W}_0(\mathcal{L}) \) which we will consider here and subsequent sections. Analogously, we also consider \( I_0(\mathcal{L}) \) and \( \mathcal{V}_0(\mathcal{L}) \); here we do not need the assumption of disjunctiveness on \( \mathcal{L} \), and \( \mathcal{V}_0(\mathcal{L}) = \{ \mathcal{V}_0(L) / L \in \mathcal{L} \} \), where \( \mathcal{V}_0(A) = \{ \mu \in I_0(\mathcal{L}) : \mu(A) = 1 \} \), \( A \in \mathcal{A}(\mathcal{L}) \). \( \mathcal{V}_0(\mathcal{L}) \) is prime complete, and is a base for the closed sets \( \mathcal{V}_0(\mathcal{L}) \) of \( I_0(\mathcal{L}) \).

Theorem 2.1 a). Consider \( I_R(\mathcal{L}) \) and \( \mathcal{W}_0(\mathcal{L}) \) with \( \mathcal{L} \) disjunctive. \( \mathcal{W}_0(\mathcal{L}) \) is regular iff for all \( \mu_1, \mu_2 \in I(\mathcal{L}) \) and \( \forall \mathcal{G}_R(\mathcal{L}) \), if \( \mu_1 \leq \mu_2 \) and \( \mu_1 \leq \nu \) then \( \mu_2 \leq \nu \).

b). The topological space \( I_R(\mathcal{L}), \mathcal{V}_0(\mathcal{L}) \) with \( \mathcal{L} \) disjunctive is considered. Then the space is \( I \) iff for \( \mu \in I(\mathcal{L}) \) and \( \mu \leq \nu \), \( \mu \leq \nu \) of \( I_R(\mathcal{L}) \) it follows that \( \nu = \nu' \).
c). Consider $I_G(L)$ and $\mathcal{V}_G(L)$. $\mathcal{V}_G(L)$ is regular iff for all $\mu_1, \mu_2 \in I(L)$ and $\nu \in \mathcal{V}_G(L)$ if $\mu_1 \leq \mu_2(L)$ and $\mu_1 \leq \nu(L)$ then $\mu_2 \leq \nu(L)$.

d). Consider the topological space $I_G(L)$, $\mathcal{V}_G(L)$. This space is $T_2$ iff for $\mu \in I(L)$ with $\mu \leq \nu_1(L)$ and $\mu \leq \nu_2(L)$ where $\nu_1, \nu_2 \in I_G(L)$, it follows $\nu_1 = \nu_2$.

Proof. The proofs for a) and c) and for b) and d) are similar. We just prove a) and b).

a). Let $\mu_1, \mu_2 \in I(L)$ such that $\mu_1 \leq \mu_2(L)$. Then there exist $\mu_1', \mu_2' \in I(\mathcal{V}_G(L))$ and $\mu_1'(\mathcal{V}_G(L)) = \mu_1(L)$, $\mu_2'(\mathcal{V}_G(L)) = \mu_2(L)$ for all $L \in I(L)$. $\mu_1(L) \leq \mu_2(L) \Rightarrow \mu_1' \leq \mu_2'$ on $\mathcal{V}_G(L)$.

Suppose $\mathcal{V}_G(L)$ is regular. Then $S(\mu_1') = S(\mu_2')$ where

$$S(\mu_1') \cap \{ \mathcal{W}_G(L) \in \mathcal{V}_G(L) : \mu_1'(\mathcal{W}_G(L)) = 1, L \in I(L) \}$$

Let now $\nu \in \mathcal{W}_G(L)$ with $\mu_1 \leq \nu$. We have $\nu \in S(\mathcal{V}_G(L))$ and $\mu_1' \leq \nu'$ on $\mathcal{V}_G(L)$, therefore $S(\nu') \subseteq S(\mu_1') = S(\mu_2')$; hence $\mu_2' \leq \nu'$ on $\mathcal{V}_G(L)$ i.e. $\mu_2 \leq \nu$ on $L$.

Conversely, let $\mu_1, \mu_2 \in I(L)$ and $\nu \in \mathcal{W}_G(L)$ such that if $\mu_1 \leq \mu_2(L)$ and $\mu_1 \leq \nu(L)$ then $\mu_2 \leq \nu(L)$. Let now $\lambda_1, \lambda_2 \in I(\mathcal{V}_G(L))$ and assume $\lambda_1 \leq \lambda_2$ on $\mathcal{V}_G(L)$. Then $\lambda_1 \leq \mu_1'$ and $\lambda_2 = \mu_2'$ where $\mu_1', \mu_2' \in I(L)$ and $\mu_1' \leq \mu_2'(\mathcal{V}_G(L))$ i.e. $\mu_1 \leq \mu_2(\mathcal{V}_G(L))$.

Now $S(\mu_2') \subseteq S(\mu_1')$. If $\lambda \in S(\mu_1')$, then clearly $\lambda \in S(\mu_2')$ and $\mu_1 \leq \lambda(L)$. Hence by the assumption $\mu_2 \leq \lambda(L)$ which implies $\lambda \in S(\mu_2')$.

b) Suppose $\mathcal{V}_G(L)$, $\mathcal{W}_G(L)$ is $T_2$, which implies that $\mathcal{V}_G(L)$ is $T_2$, and let $\mu', \nu_1, \nu_2$ as above. Then $\mu' \leq \nu'(L)$ on $\mathcal{V}_G(L)$ where $\mu' \in I(\mathcal{V}_G(L))$ and $\nu' \in \mathcal{W}_G(L)$, which implies $\nu \leq \nu'(\mathcal{V}_G(L))$. Also $\mu' \leq \nu'$ on $\mathcal{V}_G(L)$ where $\nu' \in \mathcal{W}_G(L)$ where $\nu' \leq \nu'(\mathcal{V}_G(L))$, which implies $\nu \leq \nu'(\mathcal{V}_G(L))$. Recall that $\mathcal{V}_G(L)$ is $T_2$ iff for each $\mu \in I(L)$, $S(\mu') = \emptyset$ or a singleton, hence since $\mathcal{V}_G(L)$ is $T_2$ it follows $\nu_1 = \nu_2$.

Conversely, assume that for $\mu \in I(L)$ and $\nu_1, \nu_2 \in \mathcal{V}_G(L)$, if $\mu \leq \nu_1(L)$ and $\mu \leq \nu_2(L)$ then $\nu_1 = \nu_2$. Suppose $S(\mu') \neq \emptyset$, where $\mu \in I(L)$, $\lambda \in I(\mathcal{W}_G(L))$ and $\lambda \neq \mu'$. If $\nu_1, \nu_2 \in S(\mu')$ then $\mu \leq \nu_1(L)$ and $\mu \leq \nu_2(L)$ i.e. $\nu_1 = \nu_2$. Therefore $\mathcal{V}_G(L)$ is $T_2$ and thus $\mathcal{V}_G(L)$ is $T_2$. 

Theorem 2.2 Consider $I^+_G(L)$ and $U^+_G(L)$. $U^+_G(L)$ is regular if

iff $I^+_G(L) = I^+_R(L)$.

Proof. Suppose $I^+_G(L) = I^+_R(L)$. Then $U^+_G(L) = W^+_G(L)$. Now let

$\mu_1, \mu_2 \in I^+_G(L)$, $\nu \in I^+_G(L)$ and $\mu_1 \preceq \mu_2 (L)$, $\mu_1 \preceq \nu (L)$. Then,

since $I^+_G(L) = I^+_R(L)$, $\mu_1 \in I^+_R(L)$ so $\mu_1 = \mu_2$ and $\mu_1 \preceq \nu$.

Conversely, suppose $U^+_G(L)$ is regular and let $\mu \in I^+_G(L)$; there

exists $\nu \in I^+_R(L)$ such that $\mu \preceq \nu (L)$ i.e. $\mu' \preceq \nu' (W^+_G(L))$, where

$\mu', \nu' \in I^+_G(V^+_G(L))$. But $S(\mu') = S(\nu')$ since $V^+_G(L)$ is regular. Hence

$\mu \in S(\nu')$ i.e. $\nu \preceq \mu (L)$. It follows $\mu = \nu$ and then $\mu \in I^+_R(L)$.

3. ON NORMAL, SLIGHTLY NORMAL, MILDLY NORMAL AND LINDELÖF LATTICES

In this section we wish to consider normality and related questions as well as Lindelöf properties concerning the lattices $W^+_G(L)$ in $I^+_R(L)$.

Definition 3.1

a) $L$ is slightly normal if for all $\mu \in I^+_G(L)$, there exists a unique

$\nu \in I^+_R(L)$ such that $\mu \preceq \nu (L)$.

b) $L$ is mildly normal if for all $\mu \in I^+_G(L)$, there exists a unique

$\nu \in I^+_R(L)$ such that $\mu \preceq \nu (L)$.

c) $L$ is almost countably compact if $\mu \in I^+_R(L')$ implies $\mu \in I^+_G(L')$.

Theorem 3.1 Suppose $L$ is disjunctive. Then

a) Consider $I^+_R(L)$ and $W^+_G(L)$ and suppose $L$ is Lindelöf and satisfies the condition: for all $\mu_1, \mu_2 \in I^+_G(L)$ and $\nu \in I^+_R(L)$, if

$\mu_1 \preceq \mu_2 (L)$ and $\mu_1 \preceq \nu (L)$, then $\mu_2 \preceq \nu (L)$. Then $W^+_G(L)$ is

slightly and mildly normal.

b) If $L$ is complement generated then $W^+_G(L)$ is slightly normal.

c) If $L$ is almost countably compact and mildly normal then

$W^+_G(L)$ is normal.

Proof. a) $L$ disjunctive and Lindelöf implies $W^+_G(L)$ Lindelöf

Also, by Theorem 2.1 it follows that $W^+_G(L)$ is regular. Then

$W^+_G(L)$ is slightly and mildly normal (see [4]).

b) $L$ complement generated implies $L = \bigcap L_n$, $L$ and $L_n \in L$, all n.

$W^+_G(L) = W^+_G(L_n) \cap \cap W^+_G(L_n) = W^+_G(L_n)$'. Hence $W^+_G(L)$ complement

generated which implies $W^+_G(L)$ slightly normal (see [4]).
c) By the assumption, for any $\mu \in I_{R}(L')$, it follows $\mu \in I_{G}(L)$ and then there exists a unique $\nu \in I_{R}(L)$ such that $\mu \leq \nu$ $(L)$. Let $\mu \in I_{G}(L)$ such that $\mu \leq \lambda (L)$ with $\lambda \in I_{R}(L')$. Also $\lambda \in I_{G}(L)$ and $\lambda \leq \mu \leq \nu$, on $L$ with $\nu \in I_{R}(L)$, unique. Therefore if $\mu \leq \nu_1 (L)$ with $\nu_1 \in I_{R}(L)$ then $\lambda \leq \mu \leq \nu_1 (L)$, and so $\nu_1 = \nu_2$. Hence $L$ is normal and also $U_{G}(L)$ is normal.

Remark. Consider $I_{G}(L)$ and $U_{G}(L)$ with $L$ Lindelöf. If for all $\mu_1, \mu_2 \in I_{G}(L)$ and $\nu \in I_{G}(L)$ such that if $\mu_1 \leq \nu_2 (L)$ and $\mu_1 \leq \nu (L)$ it follows that $\mu_2 \leq \nu_2 (L)$, then $U_{G}(L)$ is slightly and mildly normal.

Proof. Similar to a) of Theorem 3.1.

We next consider the following condition:

1. For any $T \in \mathcal{F}_{G}(L)$, there exists $\nu \in I_{G}(L)$ such that $T \leq \nu (L)$.

Theorem 3.2.

1. If condition (1) is satisfied and if L is prime complete then L is Lindelöf.

2. If L is Lindelöf then condition (1) holds.

3. L satisfies condition (1) iff $I_{G}(L)$, $\mathcal{F}_{G}(L)$ is Lindelöf.

Proof. a) Let $T \in \mathcal{F}_{G}(L)$ be an $L$-filter with the countable intersection property. By condition (1) there exists $\nu \in I_{G}(L)$ and $T \leq \nu (L)$ prime complete implies $S(T) \neq \emptyset$ and then $S(T) \neq \emptyset$.

b) Let $T \in \mathcal{F}_{G}(L)$. Since L is Lindelöf, $S(T) \neq \emptyset$ and therefore there exists $x \in X$ such that $x \in S(T)$. Then $T \leq x (L)$ and $\mu \in I_{G}(L)$.

c) Suppose that L satisfies condition (1). Let $T \in \mathcal{F}_{G}(L)$ and define $T(L) = T(V_{G}(L))$, $L \in L$. If $L \nsubseteq \emptyset$, $L \in L$ then $V_{G}(L) \nsubseteq \emptyset$ and $T(V_{G}(L)) \rightarrow 0$, i.e., $T \in \mathcal{F}_{G}(L)$. By condition (1), there exists $\nu \in I_{G}(L)$ such that $T \leq \nu (L)$. Hence $\nu \in I_{G}(L)$ and $T \leq \nu$ on $U_{G}(L)$, where $\nu(V_{G}(L)) = \nu(L)$. Therefore $U_{G}(L)$ satisfies condition (1). Next, we show that $U_{G}(L)$ is prime complete. For this, let $S(\nu') = \bigcap_{L \in L} \nu(L) \in V_{G}(L)$. But
\[ \gamma'(V_\mathcal{G}(L)) = 1 \iff \mathcal{G}(L) = \{ \mu \in \mathcal{L} \mid \mu(L) = 1, L \in \mathcal{L} \}. \]

Hence \( V_\mathcal{G}(L) \neq \emptyset \) which implies \( S(\gamma') \neq \emptyset \). Now, \( V_\mathcal{G}(L) \) satisfies condition (1) and prime complete implies \( V_\mathcal{G}(L) \) Lindelöf and then \( \mathcal{G}(L) \) is Lindelöf.

Conversely, let \( (V_\mathcal{G}(L), \mathcal{G}(L)) \) be Lindelöf. Let \( \mathcal{G}(L) \) and define \( \gamma'(V_\mathcal{G}(L)) = \mathcal{G}(L), L \in \mathcal{L}. \) Then \( V_\mathcal{G}(L) \) implies \( L \neq \emptyset \) and \( \mathcal{G}(L) = \mathcal{G}(L) \rightarrow 0 \), hence \( \mathcal{G}(L) \) and \( \mathcal{G}(L) \) Lindelöf implies \( \mathcal{G}(L) \) Lindelöf and then \( \mathcal{G}(L) \) satisfies condition (1); hence there exists \( \gamma \in \mathcal{G}(V_\mathcal{G}(L)) \) such that \( \gamma \leq \gamma' (V_\mathcal{G}(L)) \), where \( \gamma'(V_\mathcal{G}(L)) = \gamma(L), L \in \mathcal{L} \). \( \gamma(L) = 1 \) implies \( \gamma'(V_\mathcal{G}(L)) = 1 \) and then \( \gamma(V_\mathcal{G}(L)) = 1 \) i.e. \( \gamma(L) = 1, L \in \mathcal{L} \). Hence \( \gamma \leq \gamma'(L) \).

\section*{4. ON PRIME COMPLETE AND COUNTABLY COMPACT LATTICES}

In this section we investigate the equivalence and consequences of stronger lattice completeness assumption.

\textbf{Theorem 4.1} Let \( \mathcal{L} \) be a disjunctive lattice. \( \mathcal{G}(L) \) is prime complete iff for \( \mu \in \mathcal{G}(L) \) there exists \( \gamma \in \mathcal{G}(L) \) such that \( \mu \leq \gamma(L) \).

Proof. Let \( \mu \in \mathcal{G}(L) \) and the associated \( \mu' \) defined by \( \mu'(V_\mathcal{G}(L)) = \mu(L), L \in \mathcal{L}. \) If \( \mathcal{G}(L) \) is prime complete, \( \gamma \in \mathcal{G}(L) \) and then there exists \( \gamma \in \mathcal{G}(\mu'), \gamma \in \mathcal{G}(L) \) and it follows that \( \mu \leq \gamma(L) \). Conversely, let \( \mu \in \mathcal{G}(V_\mathcal{G}(L)) \) and consider the associated \( \mu \in \mathcal{G}(L) \) such that \( \mu'(V_\mathcal{G}(L)) = \mu(L). \) For \( \mu \in \mathcal{G}(L) \), there exists \( \gamma \in \mathcal{G}(L) \) such that \( \mu \leq \gamma(L) \). Therefore \( \gamma \in \mathcal{G}(\mu') \) and \( \mu \leq \gamma'(V_\mathcal{G}(L)) \) which implies \( \gamma \in \mathcal{G}(L) \) and since \( \mathcal{G}(L) \) is replete, \( \gamma \in \mathcal{G}(L) \).

\textbf{Theorem 4.2}

a) Let \( \mathcal{L} \) be disjunctive, almost countably compact and mildly normal and let \( V_\mathcal{G}(L) \) be prime complete. Then \( \mathcal{L} \) is countably compact.

b) Let \( \mathcal{L} \) be disjunctive, regular, Lindelöf, almost countably compact and let \( V_\mathcal{G}(L) \) be prime complete. Then \( \mathcal{L} \) is countably compact.
Proof. a) Must show that $I_R(L) = I_{R^L}(L)$. Let $\mu \in I_R(L)$; we have $\mu \leq \gamma(L')$ where $\gamma \in I_R(L')$. Since $L$ is almost countably compact we have $\gamma \leq \mu(L)$ with $\mu \in I_R(L)$ and $\gamma \in I_0(L)$. But $\gamma \in I_0(L)$ is prime complete and by Theorem 4.1 there exists $\gamma \in I_0(L)$ such that $\gamma \leq \gamma(L)$.

L almost countably compact and mildly normal implies $L$ normal (see [4]). By the normality of $L$ the $L$-regular measure $\mu$ such that $\gamma \leq \mu$ must be unique, hence $\mu = \gamma \in I_0(L)$.

b) $L$ regular and Lindelöf implies $L$ mildly normal and by the above result, it follows that $L$ is countably compact.

Theorem 4.3 Suppose $I_0(L)$, $\gamma(L)$ is $T_1$ and $L$ is disjunctive and $\gamma(L)$ prime complete. Then $I_0(L) = I_{R^L}(L)$.

Proof. Since $I_0(L)$, $\gamma(L)$ is $T_1$, given $\mu_1, \mu_2$ with $\mu_1, \mu_2 \in I_0(L)$, there exist $L_1, L_2 \in L$ such that $\mu_1 \in \gamma(L_1), \mu_2 \in \gamma(L_1)$ and $\mu_1 \in \gamma(L_2), \mu_2 \in \gamma(L_2)$. Therefore $\mu_1(L_1) = 1, \mu_2(L_1) = 0$ or $\mu_1(L_1) = 0, \mu_2(L_1) = 1$ and $\mu_1(L_2) = 1, \mu_2(L_2) = 0$ or $\mu_1(L_2) = 0, \mu_2(L_2) = 1$.

Since $\gamma(L)$ prime complete, given $\mu \in I_0(L)$ there exists $\gamma \in I_0(L)$ with $\gamma \leq \gamma(L)$, i.e. $\mu \leq \gamma$, by above there exists $L \in L$ such that $\gamma(L) = 0$ and $\mu(L) = 1$.

This is a contradiction, hence $\mu = \gamma$, and $I_0(L) = I_{R^L}(L)$.

Definition 4.1 Let $\mu \in I(L), E \subseteq X$ and define

$$
\mu^*(E) = \inf \left\{ \mu \left( L_1 \right), E \subseteq \bigcup_{i=1}^{L_1}, L_i \in L \right\} = \inf \left\{ \mu \left( L' \right), E \subseteq L', L_1 \in L \right\}.
$$

Definition 4.2 Let $\mu \in I_0(L), E \subseteq X$ and define

$$
\mu''(E) = \inf \left\{ \mu \left( L_1 \right), E \subseteq \bigcup_{i=1}^{L_1}, L_i \in L \right\}.
$$

Clearly, $\mu'$ is a finitely subadditive outer measure and $\mu''$ is an outer measure (see [7]). Let $\mu''$ be the set of $\mu''$-measurable sets, where $E$ is measurable with respect to $\mu''$ if for any

$$
A \subseteq X, \mu''(A) = \mu''(A \cap E) + \mu''(A \cap E').
$$
Theorem 4.4

Let \( \mu \in \mathcal{I}(\mathcal{L}) \). Suppose \( \mathcal{L} \subseteq \mathcal{F}^\mu \) and \( \mathcal{L} \) semiseparates \( T(\mathcal{L}) \). Then \( \mu \in \mathcal{F}^\mu(\mathcal{L}) \) and \( \mu^\prime \mid \mathcal{A}(\mathcal{L}) \in \mathcal{I}^\mu(\mathcal{L}) \).

Proof. Let \( \mu \in \mathcal{I}(\mathcal{L}) \). Then we have \( \mathcal{L} \subseteq \mathcal{F}^\mu \) and \( \mathcal{L} \subseteq \mathcal{F}^\mu \) which is closed under complement and countable unions (see [7]). Therefore \( \mathcal{A}(\mathcal{L}) \subseteq \mathcal{F}^\mu \). \( \mu^\prime \mid \mathcal{A}(\mathcal{L}) \) is then a measure on \( \mathcal{A}(\mathcal{L}) \).

\( \mu^\prime \) countably additive implies \( \mu^\prime \mid \mathcal{A}(\mathcal{L}) \in \mathcal{I}(\mathcal{L}) \). To show that \( \mathcal{L} \), assume \( \mu^\prime(\mathcal{A}')=1 \), \( \mathcal{A} \in \mathcal{L} \). Then there exist \( \{L_n\} \), \( L_n \in \mathcal{L} \) such that \( \mathcal{A} \supseteq \bigcap L_n \) and \( \mu(L_n)=1 \) for all \( n \).

But \( \bigcap L_n \in \mathcal{L} \) and \( \mathcal{A} \cap (\bigcap L_n)=\emptyset \). Hence by semiseparation there exists \( L \in \mathcal{L} \) such that \( \mathcal{A} \cap L=\emptyset \, \text{or} \, \mathcal{A} \cap A' \) and \( \bigcap L_n \subseteq \mathcal{L} \). May assume \( \mathcal{L} \) and then \( \mu^\prime(\bigcap L_n)=1 \). We then have \( \bigcap L_n \subseteq \mathcal{L} \subseteq \mathcal{A}' \) which implies \( \mu^\prime(\mathcal{L})=1 \), i.e. \( \mu^\prime \mid \mathcal{A}(\mathcal{L}) \in \mathcal{I}(\mathcal{L}) \).

5. STRONGLY \( \mathcal{G} \)-SMOOTH MEASURES

Here we consider another general Wallman space and analyze the relevant lattice in detail.

Definition 5.1

A measure \( \mu \in \mathcal{I}(\mathcal{L}) \) is strongly \( \mathcal{G} \)-smooth on \( \mathcal{L} \) iff for any sequence \( \{L_n\} \), \( L_n \in \mathcal{L} \), \( n, \Psi \), if \( \bigcap L_n \in \mathcal{L} \) then

\[ \mu \left( \bigcap L_n \right) = \inf_{n} \mu(L_n) = \lim_{n \to \infty} \mu(L_n). \]

We denote \( \mathcal{J}(\mathcal{L}) \) the set of strongly \( \mathcal{G} \)-smooth nontrivial zero-one valued measures on \( \mathcal{L} \).

Definition 5.2

The lattice \( \mathcal{L} \) is weakly prime complete if for \( \mu \in \mathcal{J}(\mathcal{L}) \), \( S(\mu) \neq \emptyset \).

Now define the following condition:

(2) For any \( \mathcal{F} \in \mathcal{F}'(\mathcal{L}) \) there exists \( \gamma \in \mathcal{J}(\mathcal{L}) \) such that \( \mathcal{F} \supseteq \gamma \mathcal{L}(\mathcal{L}) \).
We summarize a few notes on $\mathcal{G}$-smoothness that will be used throughout this section for the reader's convenience (see [6]).

a) $I_\sigma(\mathcal{L}) \subset J(\mathcal{L}) \subset I_\delta(\mathcal{L})$

b) $\mathcal{L}$ normal and complement generated implies $J(\mathcal{L}) \subset I_\delta(\mathcal{L})$

c) $\mu \in I_\sigma(\mathcal{L})$ and $\mu' \in \mu''(\mathcal{L'})$ implies $\mu \in J(\mathcal{L})$.

d) Since $\mu \in I_\sigma(\mathcal{L})$ implies $\mu' = \mu''(\mathcal{L'})$, it follows that $\mu \in J(\mathcal{L})$ and then $I_\sigma(\mathcal{L}) \subset J(\mathcal{L})$.

Theorem 5.1

a) If condition (2) holds and if $\mathcal{L}$ is weakly prime complete then $\mathcal{L}$ is Lindelöf.

b) If $\mathcal{L}$ is Lindelöf then condition (2) holds.

Proof. Omitted.

Theorem 5.2 Define $\mathcal{V}_j(\mathcal{L}) = \{ \mathcal{V}_j(L) / L \in \mathcal{L} \}$ where

$v_j(L) = \{ \mu \in J(\mathcal{L}) / \mu(L) = 1, L \in \mathcal{L} \}$. Then $\mathcal{L}$ satisfies condition (2) if $\mathcal{V}_j(\mathcal{L})$ is Lindelöf.

Proof. Suppose $\mathcal{L}$ satisfies condition (2). We show that $\mathcal{V}_j(\mathcal{L})$ satisfies condition (2). For this, let $\forall \mathcal{V}_j(\mathcal{L})$ and define $\forall(L) = \forall(v_j(L)), L \in \mathcal{L}$. If $L_n \not\in \mathcal{L}$ then $v_j(L_n) \not\in \mathcal{L}$ and $\lim n(L_n) = \lim n(v_j(L_n)) = 0$, hence $\forall \in \mathcal{V}_j(\mathcal{L})$.

By condition (2) there exists $\forall \in \mathcal{V}_j(\mathcal{L})$ such that $\forall \leq v_j(\mathcal{L})$. Hence $\forall \in \mathcal{V}_j(\mathcal{L})$ and $\forall \mathcal{V}_j(\mathcal{L})$ where $\forall(v_j(L)) = \forall(L), L \in \mathcal{L}$. For $\forall \in \mathcal{V}_j(\mathcal{L})$, there exists $\forall \mathcal{V}_j(\mathcal{L})$ such that

Next we show that $\mathcal{V}_j(\mathcal{L})$ is weakly prime complete, let

$\forall(\mathcal{V}_j(L)) = \bigcup \{ v_j(L) / \forall(v_j(L)) = 1, L \in \mathcal{L} \}$. $\forall(\mathcal{V}_j(L)) = 1$ if $\forall(\mathcal{V}_j(L)) = 1$ if $\forall \in v_j(L)$ where $\bigcup \{ \mu \in J(\mathcal{L}) / \mu(L) = 1, L \in \mathcal{L} \}$.

Hence $\forall(v_j(L)) = \forall(L)$ implies $\forall(\mathcal{V}_j(L)) = \forall(L)$. Therefore $\forall \in \mathcal{V}_j(\mathcal{L})$ is Lindelöf, and then $\mathcal{V}_j(\mathcal{L})$ is Lindelöf.

Conversely, assume (J(\mathcal{L}), $\mathcal{V}_j(\mathcal{L})$) is Lindelöf and let $\forall \in \mathcal{V}_j(\mathcal{L})$.

Define $\forall'(v_j(L)) = \forall(L), L \in \mathcal{L}$.

Then $\forall'(v_j(L))$ which implies $\forall' \mathcal{V}_j(\mathcal{L})$ and $\lim n(L) = \lim n(v_j(L)) = 0$ i.e. $\forall' \in \mathcal{V}_j(\mathcal{L})$.

Thus $\forall \mathcal{V}_j(\mathcal{L})$ Lindelöf, then $\forall \mathcal{V}_j(\mathcal{L})$ Lindelöf, then $\forall \mathcal{V}_j(\mathcal{L})$ satisfies condition (2). Hence there exists $\forall \mathcal{V}_j(\mathcal{L})$ such that $\forall \mathcal{V}_j(\mathcal{L})$ on $\forall \mathcal{V}_j(\mathcal{L})$, where $\forall'(v_j(L)) = \forall(L), L \in \mathcal{L}$. Therefore $\forall \mathcal{V}_j(\mathcal{L})$. 


Theorem 5.3 Consider \( J(L) \) and \( V_j(L) \). \( V_j(L) \) is regular iff for all \( \mu_1, \mu_2 \in \mathcal{I}(L) \) and \( \nu \in J(L) \), if \( \mu_1 \leq \mu_2 \) and \( \mu_1 \leq \nu_j(L) \) then \( \mu_2 \leq \nu_j(L) \).

Proof. For \( \mu_1, \mu_2 \in \mathcal{I}(L) \) we have \( \mu_1, \mu_2 \in \mathcal{I}(L) \) and then \( \mu_1, \mu_2 \in \mathcal{I}(V_j(L)) \), \( \mu_1(V_j(L)) = \mu_1(L) \) and \( \mu_2(V_j(L)) = \mu_2(L) \). If \( V_j(L) \) is regular then \( S(\mu_1) = S(\mu_2) \), where \( S(\mu_1) = \{ \nu_j(L) \in V_j(L) : \mu_1(V_j(L)) = 1 \} \). Let \( \nu \in J(L) \); \( \nu \in \mathcal{I}(V_j(L)) \) and \( \mu_1 \leq \nu \) on \( V_j(L) \).

Then \( \nu \in S(\mu_1) \) i.e. \( \mu_2 \leq \nu \) (L).

Conversely, suppose \( \mu_1, \mu_2 \in \mathcal{I}(L) \) and \( \nu \in J(L) \) such that if \( \mu_1 \leq \mu_2 \) and \( \mu_2 \in \mathcal{I}(V_j(L)) \) and \( \mu_1 \leq \nu \) on \( V_j(L) \). Then \( \lambda_1 = \mu_1 \) and \( \lambda_2 = \mu_2 \) with \( \mu_1, \mu_2 \in \mathcal{I}(L) \).

Thus \( \lambda_1 \leq \lambda_2 \) on \( V_j(L) \) which implies \( \mu_1 \leq \mu_2 \) on \( L \), hence \( S(\mu_2) \subseteq S(\mu_1) \). If \( \lambda \in S(\mu_1) \) then clearly \( \lambda \in J(L) \) and \( \mu_1 \leq \lambda \) (L).

By the condition of the statement, \( \mu_2 \leq \nu \) (L) and then \( \lambda \in S(\mu_2) \)

Hence \( S(\mu_2) = S(\mu_1) \) and \( V_j(L) \) is regular.

Theorem 5.4 Consider \( J(L) \), \( V_j(L) \). If \( V_j(L) \) is regular, then \( J(L) = \mathcal{I}_R^R(L) \).

Proof. Let \( \mu \in J(L) \). Then there exists \( \nu \in \mathcal{I}_R(L) \) such that \( \mu \leq \nu \) (L), hence \( \mu \leq \nu \) on \( V_j(L) \), where \( \nu \in J(L) \) and \( \nu \in \mathcal{I}_R(V_j(L)) \). \( V_j(L) \) regular implies \( S(\mu') = S(\nu') \), therefore \( \nu \leq \mu' \) (L). Then \( \mu = \nu \) (L) and since \( \nu \in \mathcal{I}_R(L) \), \( J(L) \subseteq \mathcal{I}_R(L) \).

It follows that \( \mu \in \mathcal{I}_R(L) \), \( \mathcal{I}_R(L) \) and then \( \mu \in \mathcal{I}_R(L) \). Thus \( J(L) = \mathcal{I}_R(L) \).

Theorem 5.5 Consider \( J(L) \) and \( V_j(L) \), with \( L \) Lindelöf.

If for all \( \mu_1, \mu_2 \in \mathcal{I}(L) \) and \( \nu \in J(L) \) such that \( \mu_1 \leq \mu_2 \) and \( \mu_1 \leq \nu \) (L) then \( \mu_2 \leq \nu \) (L) it follows that \( V_j(L) \) is slightly and mildly normal.

Proof. By Theorem 5.3 \( V_j(L) \) is regular. We show as in Remark of Theorem 3.1 that \( V_j(L) \) is Lindelöf and then, \( V_j(L) \) being regular and Lindelöf, it follows that it is also slightly and mildly normal.
REFERENCES


