THE DARBOUX-IONESCU PROBLEM FOR A THIRD ORDER SYSTEM OF HYPERBOLIC EQUATIONS

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Abstract. We state and prove an existence and uniqueness theorem for the third order Darboux-Ionescu Problem and we give some extensions.

1 Introduction

In his doctoral thesis [3] of 1927, D.V. Ionescu, considered the problems of Darboux, Cauchy, Picard and Goursat for hyperbolic differential equation with modified argument of the form

$$
\frac{\partial^2 u(x, y)}{\partial x \partial y} = \rho \left\{ a_1(x, y)u(\omega_1(x, y), \pi_1(x, y)) + b_1(x, y) \frac{\partial u(\omega_1(x, y), \pi_1(x, y))}{\partial x} + c_1(x, y) \frac{\partial u(\omega_1(x, y), \pi_1(x, y))}{\partial x} + a_2(x, y)u(\omega_2(x, y), \pi_2(x, y)) + b_2(x, y) \frac{\partial u(\omega_2(x, y), \pi_2(x, y))}{\partial x} + c_2(x, y) \frac{\partial u(\omega_2(x, y), \pi_2(x, y))}{\partial x} \right\} + f(x, y),
$$

where \(\omega_1(x, y), \pi_1(x, y), \omega_2(x, y), \pi_2(x, y)\) are continuous functions of \(x\) and \(y\) in the domain \(D \subset \mathbb{R}^2\).

The Darboux-Ionescu Problem was studied again in a more general frame by Ioan A. Rus in [4], [5], [6].

He considered the equation

$$
\frac{\partial^2 u(x, y)}{\partial x \partial y} = F(x, y, u(g(x, y), h(x, y))), \quad (x, y) \in I_2
$$

where \(I_2 = [0, a] \times [0, b]\), with the boundary value conditions

\[
\begin{align*}
    u(x, 0) &= \varphi(x), & x \in [0, a] \\
    u(0, y) &= \psi(y), & y \in [0, b]
\end{align*}
\]

where \(\varphi \in C^1[0, a]\), \(\psi \in C^1[0, b]\), \((g, h) \in C(I_2, I_2)\) and \(\varphi(0) = \psi(0) = \nu_0\).

G. Teodora also studied [8]–[12] the Darboux Problem for differential inclusions and multivalued functions.
V. Berinde [1] has obtained some new results using generalized Lipschitz conditions. B. Rzepecki [7] considered the third order hyperbolic equation

$$u_{x_1 x_2 x_3}^m = f(x_1, x_2, x_3, u, u_{x_1}^1, u_{x_2}^1, u_{x_3}^1, u_{x_1 x_2}^1, u_{x_1 x_3}^1, u_{x_2 x_3}^1),$$

with Darboux-type conditions and established an existence theorem using Sadovski's fixed point theorem.

2 Main result

We now consider the Darboux-Ionescu Problem for a third order system of equation with modified argument,

$$\begin{align*}
\frac{\partial^3 u_i(x, y, z)}{\partial x \partial y \partial z} &= F_i(x, y, z, u_1(x, y, z), g_1(x, y, z), h_1(x, y, z)), \\
&\vdots \\
&u_m(x, y, z, u_m(x, y, z), g_m(x, y, z), h_m(x, y, z)),
\end{align*}$$

for $i \in I_m$, $(x, y, z) \in I_3$ where $I_3 = [0, a] \times [0, b] \times [0, c]$, with the boundary conditions

$$\begin{align*}
\left\{ 
\begin{array}{l}
u_1(x, y, 0) = \varphi_i(x, y), \text{ if } (x, y) \in [0, a] \times [0, b]; \\
u_2(x, 0, z) = \psi_i(x, z), \text{ if } (x, z) \in [0, b] \times [0, c]; \\
u_3(0, y, z) = \chi_i(y, z), \text{ if } (y, z) \in [0, b] \times [0, c], \\
\end{array}
\right.
\end{align*}$$

for $i \in I_m$, where $\varphi_i$, $\psi_i$, and $\chi_i$ are continuous function with respect to all the variables on their domain and there are such that

$$\begin{align*}
u_1(x, 0, 0) &= \varphi_i(x, 0) = \psi_i(0, x) = \psi_i^2(x), \\
u_2(0, y, 0) &= \chi_i(y, 0) = \varphi_i(0, y) = \varphi_i^2(y), \\
u_3(0, 0, z) &= \psi_i(0, z) = \chi_i(0, z) = \psi_i^3(z), \\
u_4(0, 0, 0) &= \psi_i^4(0) = \psi_i^3(0) = \psi_i^2(0) = \psi_i(0),
\end{align*}$$

for $\forall (x, y, z) \in I_3$ and $f : I_3 \rightarrow (0, a)^m$, $g : I_3 \rightarrow (0, b)^m$, respectively $h : I_3 \rightarrow (0, c)^m$ with the components $(f_i, g_i, h_i) \in C(I_3, I_3), \quad i \in I_m$.

We shall now use the matrix-form of this system,

$$\begin{align*}
\frac{\partial^3 u(x, y, z)}{\partial x \partial y \partial z} &= F(x, y, z, u(f(x, y, z), g(x, y, z), h(x, y, z)))
\end{align*}$$

with the boundary conditions

$$\begin{align*}
\left\{ 
\begin{array}{l}
u_1(x, y, 0) = \varphi(x, y), \text{ if } (x, y) \in [0, a] \times [0, b]; \\
u_2(x, 0, z) = \psi(x, z), \text{ if } (x, z) \in [0, b] \times [0, c]; \\
u_3(0, y, z) = \chi(y, z), \text{ if } (y, z) \in [0, b] \times [0, c], \\
\end{array}
\right.
\end{align*}$$

which are such that

$$\begin{align*}
u_1(x, 0, 0) &= \varphi(x, 0) = \psi(0, x) = \psi^2(x), \\
u_2(0, y, 0) &= \chi(y, 0) = \varphi(0, y) = \varphi^2(y), \\
u_3(0, 0, z) &= \psi(0, z) = \chi(0, z) = \psi^3(z), \\
u_4(0, 0, 0) &= \psi^4(0) = \psi^3(0) = \psi^2(0) = \psi(0).
\end{align*}$$
By a solution of the problem (1)-(2) we mean a function from set \( \mathcal{K} \) of the vector-functions of three variables \( u : I_3 \rightarrow \mathbb{R}^m \), which are continuous together with the partial derivatives of the components of \( u, \ \frac{\partial u_i}{\partial x}, \ \frac{\partial u_i}{\partial y}, \ \frac{\partial u_i}{\partial z}, \ \frac{\partial^2 u_i}{\partial x^2}, \ \frac{\partial^2 u_i}{\partial x \partial y}, \ \frac{\partial^2 u_i}{\partial x \partial z}, \ \frac{\partial^2 u_i}{\partial y^2}, \ \frac{\partial^2 u_i}{\partial y \partial z}, \ \frac{\partial^2 u_i}{\partial z^2} \)
\( i \in \overline{1, m} \) and which satisfies (1)-(2).

Then we have the following existence and uniqueness theorem:

**Theorem 2.1.** We assume that

(i) \( f \in C(I_3 \times \mathbb{R}^m, \mathbb{R}^m); \ g \in C(I_3, [0, b]^m); \ h \in C(I_3, [0, c]^m); \)

(ii) \( \varphi \in C([0, a] \times [0, b]); \ \psi \in C([0, c] \times [0, a]); \ \chi \in C([0, b] \times [0, c]) \) with

\[
\varphi(x, 0) = \psi(0, x) = v^1(x), \quad \chi(y, 0) = \varphi(0, y) = v^2(y),
\]

\[
\psi(z, 0) = \chi(0, z) = v^3(z), \quad u(0, 0, 0) = v^1(0) = v^2(0) = v^3(0) = v^0,
\]

for \( \forall (x, y, z) \in I_3. \)

(iii) There exists the matrix-function \( \ell : I_3 \rightarrow \mathcal{M}_{mn}(\mathbb{R}^m) \) defined by \( \ell(x, y, z) = (\ell_{ij}(x, y, z))_{i, j \in \overline{1, m}} \) with \( \ell \in C(I_3, \mathcal{M}_{mn}(\mathbb{R}^m)). \)

such that

\[
||F(x, y, z, u) - F(x, y, z, \bar{u})|| \leq \ell(x, y, z) \cdot ||u - \bar{u}||,
\]

for \( \forall (x, y, z) \in I_3, \ u, \bar{u} \in \mathbb{R}^m \); that is, there exists the functions \( \ell_{ij} \in C(I_3), \ i, j \in \overline{1, m} \) such that

\[
|F_i(x, y, z, u_1, u_2, \cdots, u_m) - F_i(x, y, z, \bar{u}_1, \bar{u}_2, \cdots, \bar{u}_m)| \leq \ell_{i1}(x, y, z)|u_1 - \bar{u}_1| + \cdots + \ell_{im}(x, y, z)|u_m - \bar{u}_m|,
\]

for \( \forall (x, y, z) \in I_3, \ u_i, \bar{u}_i \in \mathbb{R}, \ i \in \overline{1, m} \);

(iv) if we denote \( L_{ij} = \max_{I_3} \int_0^x \int_0^y \int_0^z \ell_{ij}(r, s, t)drdsdt, \) and \( \mathcal{L} = (L_{ij})_{i, j \in \overline{1, m}} \in \mathcal{M}_{mn} \), then the matrix \( \mathcal{L} \) converges to \( \theta_m \in \mathcal{M}_{mn}. \)

Then the Darboux-Ionescu Problem (1)-(4) has a unique solution \( u^* \in C(I_3) \) and the solution can be obtained by the method of successive approximations starting from any function \( u_0 \in \mathcal{K}. \)

**Proof.** The problem (1)-(2) is equivalent to the following system of integral equations:

\[
u(x, y, z) = \int_0^x \int_0^y \int_0^z F(r, s, t, u(f(r, s, t), g(r, s, t), h(r, s, t)))drdsdt + \]

\[
+ \varphi(x, y) + \psi(x, z) + \chi(y, z) - v^1(x) - v^2(y) - v^3(z) + v^0.
\]

For the proof we shall use the generalised norm in \( \mathbb{R}^m. \)

We now introduce the operator \( A : [C(I_3)]^m \rightarrow [C(I_3)]^m \) defined by

\[
(Au)(x, y, z) = \int_0^x \int_0^y \int_0^z F(r, s, t, u(f(r, s, t), g(r, s, t), h(r, s, t)))drdsdt + \]

\[
+ \varphi(x, y) + \psi(x, z) + \chi(y, z) - v^1(x) - v^2(y) - v^3(z) + v^0
\]
for $\forall u \in C(I_3, \mathbb{R}^m)$ and $\forall (x, y, z) \in I_3$.

We have to show that the operator $A$ is a contraction:

$$\| (Au)(x, y, z) - (A\tilde{u})(x, y, z) \| \leq$$

$$\leq \int_0^s \int_0^s \int_0^s \| F(r, s, t, u(r, s, t), g(r, s, t), h(r, s, t)) -$$

$$- F(r, s, t, \tilde{u}(r, s, t), g(r, s, t), h(r, s, t)) \| dr ds dt \leq$$

$$\leq \int_0^s \int_0^s \int_0^s \| u(r, s, t) - \tilde{u}(r, s, t) \| dr ds dt \leq \| u - \tilde{u} \| \leq C \| u - \tilde{u} \|.$$

We have obtained

$$\| Au - A\tilde{u} \| \leq C \| u - \tilde{u} \|,$$

and from (v) follows that the matrix $C$ converges to the null-matrix, then it results from

the well known Perov theorem that the operator $A$ has a unique fixed point $u^*$, which is the solution of the integral equation system (2), and therefore of the problem (1)-(2).

Then we have

$$\frac{\partial^2 u^*(x, y, z)}{\partial x \partial y \partial z} = F(x, y, z, u^*(f(x, y, z), g(x, y, z), h(x, y, z))), \quad \text{if} \quad (x, y, z) \in I_3, \quad (8)$$

with the boundary conditions

$$\begin{cases} u^*(x, y, 0) = \varphi(x, y), \text{if} \quad (x, y) \in [0, a] \times [0, b], \\ u^*(x, 0, z) = \psi(x, z), \text{if} \quad (x, z) \in [0, c] \times [0, a], \\ u^*(0, y, z) = \chi(y, z), \text{if} \quad (y, z) \in [0, b] \times [0, c]. \quad (9) \end{cases}$$

3 Extensions

Let now consider the Darboux-Ionescu Problem for the system with modified argument of the form

$$\frac{\partial^2 u(x, y, z)}{\partial x \partial y \partial z} = F(x, y, z, u(f(x, y, z), g(x, y, z), h(x, z, y))), \quad (10)$$

for $(x, y, z) \in I_3$, where $I_3 = [0, a] \times [0, b] \times [0, c]$.

Let $I_i = [-\nu_i, 0] \times [-\nu_i, 0] \times [-\nu_i, 0]$, where $\nu_i > 0$ and we consider

$f : I_i \to [-\nu_i, 0]^m$, $g : I_3 \to [-\nu_i, 0]^m$, respectively $h : I_3 \to [-\nu_i, 0]^m$ with the components $(f_i, g_i, h_i) \in C(I_3, I_i)$, $i = 1, 2, 3$ and $f_i(x, y, z) \leq x$, $g_i(x, y, z) \leq y$, $h_i(x, y, z) \leq z$ for $\forall i \in I_3$.

We look for the solutions of the system (10) with the boundary conditions

$$u(x, y, z) = \varphi(x, y, z), \quad (x, y, z) \in I_{\nu} \setminus I_3, \quad (11)$$

where $\varphi \in C(I_3, \mathbb{R}^m)$.

By a solution of the problem (10)-(11) we mean a function from set $K_\nu$ of the vector-functions of three variables $u : I_{\nu} \to \mathbb{R}^m$, which are continuous in $I_{\nu}$ and the partial
derivatives of the components of $u$, \( \frac{\partial u_i}{\partial x}, \frac{\partial u_i}{\partial y}, \frac{\partial m_i}{\partial x}, \frac{\partial^2 u_i}{\partial x \partial y}, \frac{\partial^2 u_i}{\partial x \partial z}, \frac{\partial^2 u_i}{\partial y \partial z} \), and \( \frac{\partial^3 u_i}{\partial x \partial y \partial z} \), \( i \in 2, m \) are continuous in \( I_3 \), and which satisfies (10)--(11).

Then we have the following

**Theorem 3.1.** We suppose that

(i) \( F \in C(I_3 \times \mathbb{R}^m, \mathbb{R}^m) \);

(ii) \( f \in C(I_3, [v_0, a]^m), g \in C(I_3, [v_0, b]^m) \), \( h \in C(I_3, [v_0, b]^m) \) and \( f_i(x, y, z) \leq x, g_i(x, y, z) \leq y, h_i(x, y, z) \leq z \) for all \( i \in 3, m \);

(iii) \( \varphi \in C^1(I_\nu \setminus I_3) \);

(iv) There exist the matrix functions \( \ell : I_1 \to M_{mm}(\mathbb{R}^+) \) defined by \( \ell(x, y, z) = (\ell_{ij}(x, y, z))_{i,j \in 1, m} \) with \( \ell \in C(I_1, M_{mm}(\mathbb{R}^+)) \) such that 

\[
\| F(x, y, z, u) - F(x, y, z, \bar{u}) \| \leq \ell(x, y, z) \cdot ||u - \bar{u}||,
\]

for all \( (x, y, z) \in I_3, u, \bar{u} \in \mathbb{R}^m \);

(v) There exists \( \tau > 0 \) such that the matrix \( \mathcal{L} = (L_{ij})_{i,j \in 1, m} \in M_{mm} \) converges to \( \theta_m \in M_{mm} \), where we have denoted 

\[
L_{ij} = \max_{I_3} \int_v^z \int_t^s \ell_{ij}(r, s, t) \cdot e^{\int_v^z f_i(r, s, t) + g_i(r, s, t) + h_i(r, s, t) - r - v - s} dr ds dt.
\]

In these conditions the Darboux-Ionescu Problem for the system (30)--(18) has a unique solution \( u^* \in C(I_3) \) and the solution can be obtained by the method of successive approximations starting from any function \( u_0 \in \mathcal{K}_u \).

**Proof.** The problem (15)--(11) is equivalent to the following integral equations system:

\[
u(x, y, z) = \int_0^x \int_0^y \int_0^z \theta(x, y, z) F(r, s, t, u(f(r, s, t), g(r, s, t), h(r, s, t))) dr ds dt + \Phi(x, y, z),
\]

where \( \Phi : I_\nu \to \mathbb{R}^m \) is defined by

\[
\Phi(x, y, z) = \begin{cases} 
\varphi(0, y, z) + \varphi(x, 0, z) + \varphi(x, y, 0) - 
\varphi(x, 0, 0) - \varphi(0, y, 0), & \text{if } (x, y, z) \in I_3^c \\
\varphi(0, 0, z) + \varphi(0, 0, 0), & \text{if } (x, y, z) \in I_3 \setminus I_3^c \\
\varphi(x, y, z), & \text{if } (x, y, z) \in I_\nu \setminus I_3^c 
\end{cases}
\]

and \( \theta : I_\nu \to [0, 1] \) is defined by

\[
\theta(x, y, z) = \begin{cases} 
1, & \text{if } (x, y, z) \in I_3^c \\
0, & \text{if } (x, y, z) \in I_\nu \setminus I_3.
\end{cases}
\]

We define the operator \( A : C(I_3) \to C(I_3) \) by

\[
(Au)(x, y, z) = \int_0^x \int_0^y \int_0^z \theta(x, y, z) F(r, s, t, u(f(r, s, t), g(r, s, t), h(r, s, t))) dr ds dt + \Phi(x, y, z),
\]

(23)
We have to show again that \( A \) is contraction. If \((x, y, z) \in I_\mu \setminus \bar{I}_3\) then \( \theta(x, y) = 0 \), so 
\((Au)(x, y, z) - (A\tilde{u})(x, y, z) = 0 \), and if \((x, y, z) \in I_3\) then \( \theta(x, y, z) = 1 \) and de successively have for the components,

\[
| (Au)(x, y, z) - (A\tilde{u})(x, y, z) | \leq \\
\leq \int_0^s \int_0^y \int_0^z |F_1(r, s, t, u(f(r, s, t), g(r, s, t), h(r, s, t))) - \\
- F_1(r, s, t, \tilde{u}(f(r, s, t), g(r, s, t), h(r, s, t)))| \, dr \, ds \, dt \\
\leq \int_0^s \int_0^y \int_0^z \sum_{j=1}^n | \xi_j | (u_j - \tilde{u}_j) e^{-[\ell_j(\tau_0, \epsilon_0) + g_2(\tau_0, \epsilon_0) + h_2(\tau_0, \epsilon_0)]} \, dr \, ds \, dt \\
e^{-\epsilon_2(f(r, s, t) + g(r, s, t) + h(r, s, t))} dr \, ds \, dt \\
e^{-\epsilon_2 |u - \tilde{u}|} dr \, ds \, dt \\
whence, multiplying by \( e^{-\epsilon |x + y + z|} \), we obtain

\[ \| (Au)(x, y, z) - (A\tilde{u})(x, y, z) \|_B \leq C \| u - \tilde{u} \|_B. \] (15)

Since we supposed that the matrix \( L \) converges to the zero matrix, then from Perov's fixed point theorem it results that \( V \) has a unique fixed point \( u^* \) which is the solution of the problem (10)-(81).

Let now \( I_\mu = [0, \mu_\mu] \times [\tau, \mu_\tau] \times [3, \mu_3] \), where \( 0 < a < \mu_a \), \( 0 < b < \mu_b \) and \( 0 < c < \mu_c \) and we consider \( f : I_3 \to [0, \mu_\mu] \), \( g : I_3 \to [0, \mu_\tau] \) and \( h : I_3 \to [0, \mu_3] \) with the components \((f_1, g_1, h_1) \in C(I_\mu, I_\mu) \), \( \in \bar{A}_m \).

We look for the solutions of the system (10) with the boundary conditions

\[ u(x, y, z) = \varphi(x, y, z), \quad (x, y, z) \in I_\mu \setminus \bar{I}_3. \] (16)

Analogously with Theorem 7 we can establish

**Theorem 3.2.** We suppose that

(i) \( F \in C(I_3 \times \mathbb{R}^m, \mathbb{R}^m) \);

(ii) \( f \in C(I_\mu, [0, \mu_\mu] \) \), \( g \in C(I_\mu, [0, \mu_\tau] \) \), \( h \in C(I_\mu, [0, \mu_3] \) \);

(iii) \( \varphi \in C^0(I_{\bar{\mu}} \setminus I_3) \).
(iv) There exists the matrix function \( \ell : I_d \rightarrow M_{\nu\mu}(\mathbb{R}_+^\nu) \) defined by \( \ell(x, y, z) = (\ell_{ij}(x, y, z))_{i, j \leq \nu} \) with \( \ell \in \mathcal{C}(I_d), M_{\nu\mu}(\mathbb{R}_+^\nu) \) such that
\[
\| F(x, y, z, v) - F(x, y, z, \bar{u}) \| \leq \ell(x, y, z) \cdot |v - \bar{u}|, \tag{17}
\]
for \( (x, y, z, v) \in I_d, u, \bar{u} \in \mathbb{R}_+^\nu \).

(v) There exists \( \tau > 0 \) such that the matrix \( C = (L_{ij})_{i, j \leq m} \in M_{\nu\mu} \) converges to \( \theta_m \in M_{\mu\mu} \). Where we have denoted
\[
L_{ij} = \max_{k \in [1, \mu]} \int_0^\mu \int_0^\mu \int_0^\mu \ell_{ij}(r, s, t) \cdot e^{-\int_0^r (j_t(r, s, t) + j_s(r, s, t) + j_r(r, s, t) - \tau - y - z) ds} dr dt.
\]
In these conditions the Darboux-Ionescu Problem for the system (10)-(16) has a unique solution \( u^* \in \mathcal{C}(I_d) \) and the solution can be obtained by the method of successive approximation starting from any function \( u_0 \in \mathcal{K}_\mu \).

References