ABOUT SOME NEW CIRCUIT PROPERTIES FOR AN UNORIENTED MATROID

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Abstract. Having as start point the circuits \( C_1, C_2 \in C \) of an unoriented matroid \( M = (S, \mathcal{F}) \), this work establishes some new properties of the rank \( r(C_1 \cap C_2) \), through the possibilities as the set \( C_1 \cap C_2 \) could be: independent, dependent, base, circuit, or none of them.

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This paper is relating to unoriented matroids, denoted by \( M = (S, \mathcal{F}) \), as in [3] or [4]. We know that the circuits of a matroid are minimal dependent sets \( C \), which determine a class \( C \). The rank of a circuit \( C \) is defined in [3] or [4], by the relation:

\[
r(C) = |C| - 1, \quad C \in C.
\]  

(1)

From the same books, we know that, there are matroids as \( U_{2,4} \), in which some sets are neither circuits and nor bases (namely maximal independent sets). One of the most interesting axiomatic system, for a matroid, is given by circuits. More exactly, the system was introduced since 1935, in [3], by H. Whitney, the father of the quoted Combinatorics domain. Also, the famous known representant of this new theory, W.T. Tutte, considers that the circuit axioms permitted the development of the Matroids, by means of the Graphs.

Form [3] and [4], with a new proof in [2], we know the following property:

**Proposition 1.** Let \( M = (S, \mathcal{F}) \) be a matroid, having \( C \) as a circuit class. If \( C_1, C_2 \in C \) so that \( C_1 \neq C_2 \), we can neither have \( C_1 \subset C_2 \), nor \( C_2 \subset C_1 \).

The rank-function \( r \), of a matroid \( M = (S, \mathcal{F}) \), was also introduced by Whitney, in his pioneer's work [5]. Today, the rank axioms, for a matroid, are very usual (see [3] and [4]). The most important properties of the matroid rank, proved in detail, and based on the inclusion-exclusion Combinatorics principle, there are in [1], and they will be presented in the:

**Theorem 1.** Let \( M = (S, \mathcal{F}) \) be a matroid and \( r \) its rank-function. Then, the following assertions are true:

- (i) \( 0 \leq r(X) \leq |X|, \forall X \subseteq S \);
- (ii) \( X \subseteq Y \) imposes \( r(X) \leq r(Y), \forall X, Y \subseteq S \);
- (iii) \( r(X) + r(Y) \geq r(X \cup Y) + r(X \cap Y), \forall X, Y \subseteq S \).
With all these anterior facts, we could introduce our new results, in the following rows.

Proposition 2. Let \( M = (S, \mathcal{F}) \) be a matroid, having \( \mathcal{C} \) as circuit class. If \( C_1, C_2 \in \mathcal{C}, C_1 \neq C_2 \), then, the lower relation is true:

\[
r(C_1 \cap C_2) \leq \min\{|C_1| - 1, |C_2| - 1\}.
\]

(2)

Proof. With the Proposition 1, we can only have \( C_1 \cap C_2 \subseteq C_1 \), or \( C_1 \cap C_2 \subseteq C_2 \). Now, using (ii) from the Theorem, it results:

\[
r(C_1 \cap C_2) \leq r(C_1), \quad \text{or} \quad r(C_1 \cap C_2) \leq r(C_2).
\]

Because \( C_1 \) and \( C_2 \) are circuits, with (1), we find:

\[
r(C_1 \cap C_2) \leq |C_1| - 1, \quad \text{or} \quad r(C_1 \cap C_2) \leq |C_2| - 1,
\]

which immediately implies the relation (2). \( \square \)

Remarks. In the following, we raise for discussion the fact that the intersection \( C_1 \cap C_2 \) could be dependent or independent, circuit, base, or neither circuit, nor base.

1. We notice that \( C_1 \cap C_2 \) could not be a dependent set. Really, because \( C_1 \cap C_2 \subseteq C_1 \), or \( C_1 \cap C_2 \subseteq C_2 \), but \( C_1, C_2 \) as circuits, are minimal dependent sets!

2. Supposing that \( C_1 \cap C_2 \) could be a circuit, it would have with necessity, as \( C_1 \cap C_2 \equiv C_1 \), or \( C_1 \cap C_2 \equiv C_2 \), which implies either \( C_1 \equiv C_2 \) (in discord with the hypothesis \( C_1 \neq C_2 \)), or \( C_1 \subseteq C_2 \), or \( C_2 \subseteq C_1 \) (in discord with the Proposition 1).

3. If \( C_1 \cap C_2 \) is base, then \( r(C - 1 \cap C_2) = |C_1 \cap C_2| \), hence:

\[
r(C_1 \cap C_2) < |C_1|, \quad \text{or} \quad r(C_1 \cap C_2) \leq |C_1| - 1 \quad \text{(and similarly,} \quad r(C_1 \cap C_2) \leq |C_2| - 1).
\]

Supposing that \( r(C_1 \cap C_2) = |C_1| - 1 \), or \( r(C_1 \cap C_2) = |C_2| - 1 \), it follows:

\[
|C_1| = |C_1 \cap C_2| + 1, \quad \text{or} \quad |C_2| = |C_1 \cap C_2| + 1,
\]

relations which can be transformed through the inclusion-exclusion principle, in:

\[
|C_1| + 1 = |C_1 \cup C_2|, \quad \text{or} \quad |C_2| + 1 = |C_1 \cup C_2|.
\]

(3)

From the remained part, containing strict inequalities, with the same inclusion-exclusion principle, and imposing the condition \( r(C_1 \cap C_2) < |C_1| - 1 \), or \( r(C_1 \cap C_2) < |C_2| - 1 \), we have:

\[
|C_1| + 1 < |C_1 \cup C_2|, \quad \text{or} \quad |C_2| + 1 < |C_1 \cup C_2|.
\]

(4)

From where:

\[
|C_1| + 1 < |C_1 \cup C_2|, \quad \text{or} \quad |C_2| + 1 < |C_1 \cup C_2|.
\]

(5)

With (3) and (5), it occurs:

\[
|C_1| + 1 \leq |C_1 \cap C_2|, \quad \text{or} \quad |C_2| + 1 \leq |C_1 \cap C_2|.
\]

(6)
4. If $C_1 \cap C_2$ is independent set, the things are similar with 3, because $r(C_1 \cap C_2) \leq |C_1 \cap C_2|$. So:

$$r(C_1 \cap C_2) \leq |C_1 \cap C_2| < |C_1|, \text{ hence } r(C_1 \cap C_2) \leq |C_1| - 1, \text{ or } r(C_1 \cap C_2) \leq |C_2| - 1,$$

and as above, we reach the same relation (6).

5. We continue our study, in the hypothesis that $C_1 \cap C_2$ is neither circuit, nor base. Also, with the inclusion-exclusion principle, we have:

$$r(C_1 \cap C_2) < |C_1 \cap C_2| = |C_1| + |C_2| - |C_1 \cup C_2|. \quad (7)$$

In (7), we shall impose the conclusion of the Proposition 2, and so, it occurs:

$$|C_1| + 1 < |C_1 \cup C_2|, \quad \text{or} \quad |C_2| + 1 < |C_1 \cup C_2|, \quad (8)$$

as a strict inequality in comparison with (6).

From (6) and (8), we should obtain a finer analysis, if we have in view that $C_1 \cup C_2$ could be base, circuit, or neither base, nor circuit. By example, if $C_1 \cup C_2$ is base, (8) gives us the relation:

$$r(C_1) + 2 < r(C_1 \cup C_2), \quad \text{or} \quad r(C_2) + 2 < r(C_1 \cup C_2), \quad (9)$$

and the relation:

$$r(C_1) + 1 < r(C_1 \cup C_2), \quad \text{or} \quad r(C_2) + 1 < r(C_1 \cup C_2), \quad (10)$$

is obtained if $C_1 \cup C_2$ is circuit.

Also, in the situation in which $C_1 \cup C_2$ is neither base, nor circuit, with the inclusion-exclusion principle, from (6) we find:

$$|C_1 \cap C_2| < r(C_1), \quad \text{or} \quad |C_1 \cap C_2| < r(C_2). \quad (11)$$

Taking consideration of all these anterior remarks, we could now emphasize the following new result:

**Proposition 3.** Let $M = (S, \mathbf{F})$ be a matroid, having $C$ as circuit class, and $C_1, C_2 \in C$, $C_1 \neq C_2$ for which the relation (2) functions as in the Proposition 2. The next assertions are true:

(i) if $C_1 \cap C_2$ is base in $M$, or if $C_1 \cap C_2$ is independent set in $M$, then:

$$|C_1 \cup C_2| \geq \max\{1 + |C_1|, 1 + |C_2|\}; \quad (12)$$

(ii) if $C_1 \cap C_2$ is neither circuit, nor base in $M$, then:

$$|C_1 \cup C_2| > \max\{1 + |C_1|, 1 + |C_2|\}. \quad (13)$$

For (3) and (13) we can obtain many other corollaries, with an analysis on $C_1 \cup C_2$ as base, circuit, neither base nor circuit, in the matroid $M$, as in the relations (9), (10), (11).
References