A QUADRATIC FREDHOLM INTEGRAL EQUATION AND ITS SOLUTION FOR VARIOUS KERNELS

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Abstract. Consider the Fredholm integral equation

\[ \varphi(x) = 1 + \lambda \varphi(x) \int_0^1 k(x, y) \varphi(y) dy, \quad \lambda \text{ a real parameter.} \]

The solution of this equation is discussed for separable, difference and distribution kernels. Existence, uniqueness, and bifurcation questions are explored for various assumptions on the kernel.

1 Introduction

Consider the Fredholm quadratic integral equation

\[ \varphi(x) = 1 + \lambda \varphi(x) \int_0^1 k(x, y) \varphi(y) dy, \quad (1) \]

where \( \lambda \) is a parameter. Equation (1) is a generalization of the Chandrasekhar \( H \)-equation

\[ H(\mu) = 1 + \mu H(\mu) \int_0^1 \frac{\phi'\left(\frac{\mu}{\mu + \mu'}\right)}{\mu + \mu'} H(\mu') d\mu'. \quad (2) \]

Chandrasekhar used the \( H \)-function in the theory of radiative transfer [1].

We rewrite (1) as

\[ \frac{\varphi(x) - 1}{\varphi(x)} = \lambda \int_0^1 k(x, y) \varphi(y) dy \quad (3) \]

and substitute \( \psi(x) = \frac{\varphi(x) - 1}{\varphi(x)} \). Then (3) reduces to

\[ \psi(x) = \lambda \int_0^1 k(x, y) \frac{1}{1 - \psi(y)} dy. \quad (4) \]

Equation (4) is a Hammerstein equation that has the general form

\[ \psi(x) + \int_0^1 k(x, y) f(y, \psi(y)) dy = 0 \quad (5) \]
where $\psi$ is the unknown and $f$ is a nonlinear function. For a discussion of equations of Hammerstein type see Tricomi [7] or Corduneanu [3]. Dolph [4] has a treatment of nonlinear equations of Hammerstein type. The results of Dolph's paper are summarized by Corduneanu [3].

In contrast with linear theory, equation (1) is a quadratic Fredholm equation of the second kind. We also discuss the solution of a quadratic Fredholm equation of the first kind

$$ g(x) = \lambda \psi(x) \int_0^1 k(x, y) \psi(y) dy $$

where $g$ is a known function and $\psi$ is unknown. The solution of the integral equation (1) is treated under various assumptions on the kernel and a bifurcation analysis discussing existence and uniqueness for the parameter $\lambda$ is presented in Section 2. In Section 3 solutions of (1) are presented with various assumptions on the kernel, and in Section 4 various generalizations of equation (1) are given. A knowledge of standard material on linear integral equations of Fredholm types with separable kernels is assumed as presented in Cochran [2].

2 Bifurcation

Consider the simple case of (1) where $k(x, y) \equiv 1$, that is, assume

$$ \psi(x) = 1 + \lambda \psi(x) \int_0^1 \psi(y) dy. $$

First, we note that solutions exist provided $\lambda \leq \frac{1}{4}$. For example, if we look for a constant solution, then we obtain

$$ \psi = 1 + \lambda \psi^2. $$

The discriminant of $\lambda \psi^2 - \psi + 1 = 0$ is equal to $1 - 4\lambda$. If we require $1 - 4\lambda \geq 0$ in order to obtain a real solution, then $\lambda \leq \frac{1}{4}$. In that case, we obtain

$$ \psi = \frac{1 \pm \sqrt{1 - 4\lambda}}{2\lambda}. $$

From (8) we solve for $\lambda$ to get

$$ \lambda = \frac{\psi - 1}{\psi^2}. $$

Figure 1 gives a graph of $\lambda$ in terms of $\psi$. Note that the bifurcation curve can be interpreted as follows: There is no real solution of (8) for $\psi$ when $\lambda > \frac{1}{4}$, and when $\lambda = \frac{1}{4}$ there is one solution $\psi = 2$. For $\lambda = 0$, we obtain the unique solution $\psi = 1$. For $0 < \lambda < \frac{1}{4}$ and for $\lambda < 0$ there are two solutions. Tricomi [6] gives a bifurcation analysis of the quadratic integral equation

$$ \psi(x) - \lambda \int_0^1 \psi^2(y) dy = 1, $$

(11)
where the bifurcation curve is the same as the one we have for (7). If we let $k(x, y) = \delta(y - x)$ in (1) we also obtain a quadratic equation where $\delta$ denotes the delta distribution. A real solution exists in this case if and only if $\lambda \leq \frac{1}{4}$.

\[\lambda = \frac{1}{4}\]

\[\varphi\]

Fig. 1.

Consider a generalization of (7) in the form

\[\varphi(x) = 1 + \lambda \varphi(x) \int_0^1 \varphi(y) dy.\]  \hspace{1cm} (12)

In order to look for a constant solution of (12) let $\varphi = K$ to obtain

\[K = 1 + \lambda K^3.\]  \hspace{1cm} (13)

Then

\[\lambda = \frac{K - 1}{K^3}.\]  \hspace{1cm} (14)

The graph of $\lambda$ in terms of $K$ is given in Figure 2. At the critical value $\lambda = \frac{4}{27}$ there is a double solution for $K$. The discriminant of the cubic (13) yields the critical value $\lambda = \frac{4}{27}$. The results obtained by using Cardan's method of solution, Turnbull [7], agrees with the bifurcation curve given in Figure 2.
More generally, consider a nonlinear integral equation

\[ \varphi(x) = 1 + \lambda \varphi^{n-1}(x) \int_0^1 \varphi(x) dx, \quad n \geq 2. \]  \hspace{1cm} (15)

Then the constant solution satisfies

\[ \varphi(x) = 1 + \lambda \varphi^n(x). \]  \hspace{1cm} (16)

Analyzing the bifurcation curve for (15) we have a behavior similar to that depicted in Figure 1 for \( n \) even and similar to that in Figure 2 for \( n \) odd. If we solve for \( \lambda \) in (16) we obtain

\[ \lambda = \frac{\varphi - 1}{\varphi^n}. \]  \hspace{1cm} (17)

The critical value of \( \lambda = \frac{(n-1)^{n-1}}{n^n} \) behaves as follows: For \( n \) even, if \( \lambda \leq \frac{(n-1)^{n-1}}{n^n} \), there is a solution of (16). For \( 0 < \lambda \leq \frac{(n-1)^{n-1}}{n^n} \), there are two solutions. For \( \lambda = 0 \), there is the unique solution \( \varphi = 1 \). For \( \lambda < 0 \), there are two solutions. If \( n \) is odd, there is a unique solution for \( \lambda \geq \frac{(n-1)^{n-1}}{n^n} \). There are three solutions for \( 0 < \lambda < \frac{(n-1)^{n-1}}{n^n} \), and a unique solution for \( \lambda \leq 0 \).

In passing, it is interesting to consider the following result regarding the complex case. Assume \( \varphi = \varphi_1 + i\varphi_2 \) where \( \varphi_1 \) and \( \varphi_2 \) are the real and imaginary parts of \( \varphi \). Substituting
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φ in (7) and equating real and imaginary parts, we obtain

\[ \varphi_1 = 1 + \lambda \left[ \varphi_1 \int_0^1 \varphi_1 - \varphi_2 \int_0^1 \varphi_2 \right] \]  \hspace{1cm} (18)

\[ \varphi_2 = \lambda \left[ \varphi_2 \int_0^1 \varphi_1 + \varphi_1 \int_0^1 \varphi_2 \right] \]  \hspace{1cm} (19)

From (18) and (19) we find

\[ \varphi_1^2 + \varphi_2^2 = \frac{1}{\lambda} \int_0^1 \varphi_2 \]  \hspace{1cm} (20)

If we let \( \alpha = \frac{1}{\lambda} \int_0^1 \varphi_2 \), then we obtain the circle \( \varphi_1^2 + \left( \varphi_2 - \frac{\alpha}{2} \right)^2 = \frac{\alpha^2}{4} \) in the \((\varphi_1, \varphi_2)\) plane.

In the simple case of a constant solution given by (9), we can directly check that \( \varphi_1^2 + \varphi_2^2 = \frac{1}{\lambda} \). Thus, for complex solutions the real and imaginary parts of \( \varphi \) are located on a circle.

3  Separable Kernels.

Next, we discuss the solution of (1) for a separable kernel

\[ k(x, y) = \sum_{i=1}^{n} A_i(x) B_i(y). \]  \hspace{1cm} (21)

First consider a special case for \( n = 1 \).

\[ k(x, y) = A(x) B(y). \]  \hspace{1cm} (22)

Substitute (22) in (1) to obtain

\[ \varphi(x) = 1 + \varphi(x) \int_0^1 A(x) B(y) \varphi(y) dy. \]  \hspace{1cm} (23)

Assume

\[ \alpha = \int_0^1 B(y) \varphi(y) dy. \]  \hspace{1cm} (24)

Substitute (24) in (23), and solve for \( \varphi(x) \) to obtain

\[ \varphi(x) = \frac{1}{1 - \lambda \alpha A(x)}. \]  \hspace{1cm} (25)

Substituting (25) in (24), we obtain

\[ \alpha = \langle \varphi, B \rangle = \left\langle \frac{1}{1 - \lambda\alpha A}, B \right\rangle \].
That is, we get
\[ \alpha = \int_0^1 \frac{B(x)}{1 - \lambda \alpha A(x)} \, dx. \] (26)

In contrast with the linear theory, equation (26) is a nonlinear equation in \( \alpha \). Even in a simple case we obtain a transcendental equation in \( \alpha \). For example, if \( B(x) = 1 \), \( A(x) = x \), we can integrate to obtain \( \lambda \alpha^2 + \ln(1 - \lambda \alpha) = 0 \). If we assume \( A(x) = \sin x \), \( B(x) = \cos x \), then we also obtain \( \lambda \alpha^2 + \ln(1 - \lambda \alpha) = 0 \).

Next we consider another special case of (21) with \( n = 2 \). If we let \( k(x, y) = \sin(x - y) \), then in (27) we let \( A_1(x) = \sin x \), \( A_2(x) = \cos x \), \( B(y) = \cos y \), \( B_2(y) = -\sin y \). Assume
\[ \alpha_1 = (B_1, \varphi) \quad \text{and} \quad \alpha_2 = (B_2, \varphi). \] (27)

Let the limits of integration in (1) be 0 and \( \frac{\pi}{2} \). Using (27) in (1) we find that \( \varphi \) has the form
\[ \varphi(x) = \frac{1}{1 - \lambda \sum_{i=1}^2 \alpha_i A_i(x)}. \] (28)

Substituting (26) in (27) we obtain
\[ \alpha_1 = \int_0^1 \frac{B_1(x)}{1 - \lambda \sum_{i=1}^n \alpha_i A_i(x)} \, dx, \quad i = 1, 2. \] (29)

In the case that \( k(x, y) = \sin(x - y) \), we obtain
\[ \alpha_1^2 + \alpha_2^2 = \ln \left( \frac{1 - \lambda \alpha_2}{1 - \lambda \alpha_1} \right). \] (30)

The latter equation can be used with (29) to approximate \( \alpha_1 \) and \( \alpha_2 \) Saaty [5].

Assume \( A_1(x) = 1 \), \( i = 1, \ldots, n \). Let \( \beta_i = \int_0^1 B_i(y) \, dy \). Then
\[ \alpha_i = \frac{1}{1 - \lambda \sum_{i=1}^n \alpha_i} \beta_i. \] (31)

Then
\[ \left( \sum_{i=1}^n \alpha_i \right) \left( 1 - \lambda \sum_{i=1}^n \alpha_i \right) = \sum_{i=1}^n \beta_i. \] (32)

We can use the quadratic equation to solve for \( \sum_{i=1}^n \alpha_i \) and then use (31) to obtain \( \alpha_i \). Note that (32) has a solution if and only if \( \lambda \leq \frac{1}{4} \).

In general, we obtain a system of nonlinear equations by substituting (21) in (1) to obtain
\[ \varphi(x) = 1 + \lambda \varphi(x) \int_0^1 \frac{\sum_{i=1}^n A_i(x)B_i(y)\varphi(y) \, dy}{1 - \lambda \sum_{i=1}^n \alpha_i A_i(x)} = \] \[ = 1 + \lambda \varphi(x) \int_0^1 \frac{\sum_{i=1}^n A_i(x)B_i(y)\varphi(y) \, dy}{1 - \lambda \sum_{i=1}^n \alpha_i A_i(x)}. \] (33)
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Define an inner product
\[ \alpha_i = \int_0^1 B_i(y)\varphi(y)dy = \langle B_i, \varphi \rangle. \]  
(34)

Using (34) in (33) we find that \( \varphi \) has the form
\[ \varphi(x) = \frac{1}{1 - \lambda \sum_{i=1}^n \alpha_i A_i(x)}. \]  
(35)

Substituting (35) in (34) we obtain
\[ \alpha_i = \langle \varphi, B_i \rangle = \left\langle \frac{1}{1 - \lambda \sum_{i=1}^n \alpha_i A_i(x)}, B_i \right\rangle, \] 
(36)

that is, we get
\[ \alpha_i = \int_0^1 \frac{B_i(x)}{1 - \lambda \sum_{i=1}^n \alpha_i A_i(x)} dx, \quad i = 1, \ldots, n. \]  
(37)

In contrast with the linear theory, the system (37) is a nonlinear system of equations in terms of \( \alpha_i, \ i = 1, \ldots, n. \)

4 Existence and Uniqueness of Solutions

Now we discuss (1) by letting \( \psi(x) = \frac{\varphi(x) - 1}{\varphi(x)} \), that reduces it to an equation of Hammerstein type. Tricomi [6] uses successive approximations to give conditions under which equations of Hammerstein type have solutions. We use equation (4) and a geometric series to give a lower bound for \( \lambda. \)

Assume \( |\psi(y)| < 1 \), so that we can use a geometric series on the right hand side of (4).

Taking the absolute value of both sides we obtain
\[ |\psi(x)| = |\lambda| \left| \int_0^1 (k(x, y) \sum_{i=0}^\infty \psi^i(y))dy \right| \]
\[ \leq |\lambda| \sum_{i=0}^\infty \int_0^1 |k(x, y)||\psi(y)|dy \]  
(38)

Let \( K(x) = \|k(x, y)\| \) with respect to \( y \). Then, by the Cauchy-Schwartz inequality,
\[ |\psi(x)| \leq \lambda \sum_{i=0}^\infty K(x) \|\psi\|^i, \]  
(39)

\( K(x) = \|k(x, y)\| \) with respect to \( y \). Since \( |\psi| < 1 \), we obtain \( \|\psi\| < 1 \). Then, we use a geometric series to obtain
\[ |\psi(x)| \leq |\lambda| \frac{1}{1 - \|\psi\|}. \]
Thus, we conclude
\[ \|\psi\| \leq |\lambda| \frac{1}{1 - \|\psi\|} K, \]  \hspace{1cm} (40)
where \( \|K(x)\| \leq K \). Let \( \beta = \|\psi\| < 1 \), then from (33) we obtain the quadratic inequality
\[ \beta^2 - \beta + |\lambda| K \geq 0. \]  \hspace{1cm} (41)
In order for \( \beta^2 - \beta + |\lambda| K \geq 0 \), we obtain
\[ 1 - 4|\lambda| K \leq 0. \]  \hspace{1cm} (42)
Thus, there is a solution \( \psi \) with the assumptions mentioned if
\[ \|\psi\| \leq \frac{1}{4K}. \]  \hspace{1cm} (43)

Our work could be extended by examining different kinds of kernels from the ones we have investigated.

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References