Cohomology of Fields of 2-jets on a Foliated Manifold

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Abstract. Let $M$ be a Riemannian foliated manifold. We define the fields of leafwise and transversal 2-jets on $M$. We obtain a relation between the basic cohomology of $M$ and the cohomology of fields of basic 2-jets. A relation between the cohomology of fields of leafwise 2-jets and the leafwise cohomology of $M$ is also obtained.

1 Preliminaries

Following [2], [3] we present some notions containing the fields of 2-jets on a manifold. Let $M$ be a $n$-dimensional manifold and denote by $\mathcal{D}(M)$ the set of differential functions on $M$.

We recall that two maps $f, g \in \mathcal{D}(M)$ determine the same 2-jet at $x \in M$ if $f(x) = g(x) = 0$ and if for every curve $\gamma : \mathbb{R} \to M$ with $\gamma(0) = x$, the curves $f \circ \gamma, g \circ \gamma$ have a second order contact at zero. If, moreover, the maps $f$ and $g$ have the first derivative in $x$ equal to zero, then they determine the same homogeneous 2-jet at $x$.

The 2-jet of $f$ at $x$ is denoted by $j^2_x f$. Obviously, $j^2_x f$ depends on the germ of $f$ at $x$ only.

Let $(U, \phi), (\bar{U}, \bar{\phi})$ be two local charts with $(x^1, \ldots, x^n), (\bar{x}^1, \ldots, \bar{x}^n)$ as local coordinates, respectively.

More generally a 2-jet at $x \in M$ is a combination locally given by

$$\omega^2 = \omega_i (x) \cdot j^2_x x^i + \frac{1}{2} \omega_{ij} (x) \cdot j^2_x x^i \cdot j^2_x x^j$$

with $\omega_{ij} = \omega_{ji}$ and where the coefficient functions are satisfying the following conditions in $U \cap \bar{U}$

$$\bar{\omega}_i = \omega_j \frac{\partial x^j}{\partial \bar{x}^i}; \quad \bar{\omega}_{ij} = \omega_{jk} \frac{\partial x^k}{\partial \bar{x}^i} \frac{\partial x^j}{\partial \bar{x}^2} + \omega_{jk} \frac{\partial^2 x^j}{\partial \bar{x}^1 \partial \bar{x}^2} \quad (1.2)$$

The set of all 2-jets at $x$ is denoted by $J^2_x M$. The set $J^2 M = \bigcup_{x \in M} J^2_x (M)$ of 2-jets on $M$ is a fiber bundles over $M$ with the fiber of dimension $\frac{n(n + 3)}{2}$ and we denote by $J^2(M)$
its sections, namely the space of fields of 2-jets on $M$.

Let $j^2 : \Omega^0(M) \longrightarrow J^2(M)$ be the map which assigns to a function $f$ the field $j^2 f$ of 2-jets of $f$ on $M$. This map is called the second differential on $M$. In local coordinates we have

$$j^2 f = \frac{\partial f}{\partial x^i} j^2 x^i + \frac{1}{2} \frac{\partial^2 f}{\partial x^i \partial x^j} j^2 x^i \cdot j^2 x^j.$$

The field of 2-jets $\omega \in J^2(M)$ is called [3]:

(i) exact if $\omega = j^2 f$ for some $f \in \Omega^0(M)$;

(ii) closed if it is locally exact.

We denote by $E^2(M)$, $C^2(M)$ the space of exact, respectively closed fields of 2-jets. We have $E^2(M) \subset C^2(M)$ and the cohomology group of fields of 2-jets on $M$ is the following quotient group:

$$H^2_0(M) = C^2(M)/E^2(M) \quad (1.3)$$

It is isomorphic with the 1-dimensional cohomology group with real coefficients of $M$ ([3], theorem II).

Now, we present some notions about the foliated manifolds using [1], [7], [9]. In this paper we consider that indices take the following values: $a, b, \ldots = 1, \ldots, n; \ u, v, \ldots = n+1, \ldots, n+m; \ i, j, \ldots = 1, \ldots, n+m$.

Let $M$ be a $n+m$ dimensional manifold. A foliation $\mathcal{F}$ of dimension $m$ and codimension $n$ on $M$ is a partition $\{\mathcal{L}_a\}_{a \in \Gamma}$ of $M$ into connected $m$-dimensional submanifolds called leaves and which have the following property. For every point of $M$ there is a local chart with domain $U$ and local coordinates $(x^a, x^u)_{a, u}$, such that for each leaf $\mathcal{L}_a$ the connected components of the plaque $U \cap \mathcal{L}_a$ are defined by the equations $dx^a = 0$, for all $a = 1, \ldots, n$. Such a chart is called a distinguished chart.

The bundle of vectors tangent to leaves is the structural bundle of $\mathcal{F}$. It will be denoted by $T\mathcal{F}$ and its sections by $\mathfrak{X}(\mathcal{F})$. If $g$ is a Riemannian metric on $M$, the transversal bundle of foliation is the normal subbundle of the structural bundle with respect to $g$ and it will be denoted by $T\mathcal{F}^\perp$; the set of its sections is $\mathfrak{X}^\perp(\mathcal{F})$.

In a distinguished chart $(x^a, x^u)_{a = 1, \ldots, n; \ u = n+1, \ldots, n+m}$ there is an adapted basis

$$\left\{\begin{array}{l}
X_a = \frac{\partial}{\partial x^a} - t_a^u \frac{\partial}{\partial x^u} \frac{\partial}{\partial x^u} \end{array}\right\} \quad (1.4)$$

in the module of vector fields of $M$ such that $T\mathcal{F} = \text{span} \left( \frac{\partial}{\partial x^a} \right); T\mathcal{F}^\perp = \text{span} \left( X_a \right)$. The functions $t_a^u$ are completely determined by the orthogonality of $T\mathcal{F}$ and $T\mathcal{F}^\perp$. The dual cobasis of the basis $\left\{ X_a, \frac{\partial}{\partial x^a} \right\}$ is

$$\left\{ dx^a, \partial^u = dx^a + t_a^u dx^a \right\} \quad (1.5)$$

We recall that a function $f \in \Omega^0(M)$ is basic (foliated) if it is constant on the leaves. We denote by $\mathcal{F}$ the sheaf of germs of basic functions on $M$. 

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Let be $\Omega^{p,q}(M)$ the set of $(p,q)$-forms and $\delta_{q1} : \Omega^{p,q}(M) \rightarrow \Omega^{p,q+1}(M)$ the foliated derivative on $M$. The $(p,q)$-cohomology group $H^{p,q}(M)$ is the $q$-dimensional cohomology group of the complex $(\Omega^{p,q}(M), \delta_{q1})$. On $\Omega^0$, the foliated derivative $\delta_{q1}$ can be identified with the operator $d_F$ defined by the de Rham derivative on the leaves.

The leafwise de Rham complex $(\Omega(F), d_F)$ is the restriction to the leaves of the de Rham complex $(\Omega, d)$ of $M$, where $\Omega(F)$ is the set of differential forms on the leaves. The cohomology $H^*(F) = H^* (\Omega(F), d_F)$ is called the leafwise cohomology of $F$. It is well-known the following de Rham type theorem:

**Theorem 1.1.** [1] $H^q(F)$ and the Čech cohomology group $H^q(M, \Phi)$ with coefficients in the sheaf $\Phi$ of germs of basic functions are isomorphic.

A $(p,0)$-form closed with respect to the foliated derivative is called a basic (foliated) $p$-form. We denote by $\Omega(M/\mathcal{F})$ the space of basic forms. The basic complex of $M$ is the subcomplex $(\Omega(M/\mathcal{F}), d)$ of the de Rham complex of $M$ and the cohomology $H^*(M/\mathcal{F}) = H^* (\Omega(M/\mathcal{F}), d)$ is called the basic cohomology.

### 2 Fields of transversal 2-jets

Let $M$ be a Riemannian foliated manifold like in the previous section. For every $f \in \Omega^0(M)$ we have in a distinguished chart the following decomposition

$$df = (X_\alpha f) dx^\alpha + \frac{\partial f}{\partial u} du^\alpha$$  \hspace{1cm} (2.1)

with respect to the adapted cobasis (1.5).

For two distinguished local charts with domains $U$, $\bar{U}$ and local coordinates $(x^a, \bar{x}^b)$, $(\bar{x}^b, \bar{x}^a)$, we obtain by a direct calculation the following relations in $U \cap \bar{U} \neq \emptyset$

$$\bar{X}_\alpha = \frac{\partial x^a}{\partial \bar{x}^b} X_\alpha; \quad \bar{X}_b X_\alpha = \frac{\partial^2 x^a}{\partial \bar{x}^b \partial \bar{x}^c} X_\alpha + \frac{\partial x^a}{\partial \bar{x}^b} \frac{\partial x^c}{\partial \bar{x}^b} X_\alpha X_b \bar{X}_c f$$

We obtain that the functions $X_\alpha f$ and $\frac{1}{2} (X_{a1} X_{a2} f + X_{b1} X_{b2} f)$ verify the relations (1.2), hence

$$f^{1,2} = (X_\alpha f) j^2 x^a + \frac{1}{4} (X_\alpha X_{a1} f + X_\alpha X_{a2} f) j^2 x^a = j^2 x^a$$  \hspace{1cm} (2.2)

is a field of 2-jets on $M$ and we call it the field of transversal 2-jets of the function $f$.

More generally, a field of 2-jets (1.1) given in a distinguished chart by

$$\omega^{1,2} = \omega^a \cdot j^2 x^a + \frac{1}{2} \omega_{a1} x^a : j^2 x^a$$  \hspace{1cm} (2.3)

is called a field of transversal 2-jets on $M$. We denote by $J^2(M/\mathcal{F})$ the space of all fields of transversal 2-jets on $M$.

The map $j^{1,2} : \Omega^0(M) \rightarrow J^2(M/\mathcal{F})$ which assign to every differential function $f$ on $M$ its field of transversal 2-jets $j^{1,2} f$ is called the transversal second differential on $M$. 

Remark 2.1. Let \((M', \mathcal{F}')\) be another foliated manifold with \(\dim \mathcal{F}' = m\), \(\text{codim} \mathcal{F}' = n'\) and \(\mu : M \to M'\) a foliated map between \(M\) and \(M'\) (\(\mu\) sends leaves into leaves). This map induces a map \(\mu^2 : J^2(M') \to J^2(M)\) locally given by
\[
\mu^2(\omega'^{i j} y'^{i} + \frac{1}{2} \omega'^{i j} y'^{i} \cdot j^{2} y'^{i}) = \omega^{i j} \frac{\partial y'^{i}}{\partial x} j^{2} x^{i} + \frac{1}{2} \left( \omega^{i j} \frac{\partial^2 y'^{i}}{\partial x \partial y} + \omega'_{ij} \frac{\partial y'^{i}}{\partial x} \right) j^{2} x^{i} j^{2} x^{j}
\]
where \((x^1, ..., x^{n+m})\), \((y^1, ..., y^{n'+m'})\) are distinguished coordinates of two local charts at \(x\), \(\mu(x)\), respectively.

We denote by \(\mu^{1,2}\) the restriction of \(\mu^2\) to \(J^2(M'/\mathcal{F}')\). Taking into account that \(\mu\) is foliated, we have \(\frac{\partial y'^{a}}{\partial x^{u}} = 0\) for all \(a = 1, n'\) and \(u = n + 1, n + m\), so
\[
\mu^{1,2}(\omega'^{i j} y'^{a} + \frac{1}{2} \omega'^{i j} y'^{a} \cdot j^{b} y'^{b}) = \omega^{i j} \frac{\partial y'^{a}}{\partial x} j^{b} x^{a} + \frac{1}{2} \left( \omega^{i j} \frac{\partial^2 y'^{a}}{\partial x \partial y} + \omega'_{ij} \frac{\partial y'^{a}}{\partial x} \right) j^{b} x^{a} j^{b} x^{b}
\]
where \(a, b = 1, n'\) and \(e_1, b_1 = 1, n\). The following map is well-defined

\[\mu^{1,2} : J^2(M'/\mathcal{F}') \to J^2(M/\mathcal{F})\]

(2.4)

If \(\mu\) is a diffeomorphism, then \(\mu^{1,2}\) becomes an isomorphism.

Definition 2.1. A field of transversal 2-jets \((\Omega^e)\) is called a field of basic 2-jets if all coefficient functions \(\omega_{\alpha \beta}\), \(\omega_{\alpha \beta \gamma}\), are basic functions.

We shall denote by \(J^2_0(M/\mathcal{F})\) the set of fields of basic 2-jets on \(M\) and it is a subspace of \(J^2(M/\mathcal{F})\). It is easy to see that \(\mu^{1,2}(J^2_0(M'/\mathcal{F}')) \subset J^2_0(M/\mathcal{F})\).

Remark 2.2. If \(f\) is a basic function, then \(j^{1,2} f\) is basic and it has the following form
\[
j^{1,2} f = \frac{\partial f}{\partial x} j^{2} x^{a} + \frac{1}{2} \frac{\partial^2 f}{\partial x^a \partial x^b} j^{2} x^{a} j^{2} x^{b}.
\]

Definition 2.3. The field of basic 2-jets \(\omega \in J^2_0(M/\mathcal{F})\) is called:

(i) exact if \(\omega = j^{1,2} f\) for some \(f \in \Phi(M)\);

(ii) closed if it is locally exact.

We denote by \(E^2_0(M/\mathcal{F})\) and \(C^2_0(M/\mathcal{F})\) the spaces of exact, respectively closed fields of basic 2-jets on \(M\) and we have \(E^2_0(M/\mathcal{F}) \subset C^2_0(M/\mathcal{F})\).

We call the cohomology group of fields of basic 2-jets on \(M\) the following quotient group:
\[H^{2,2}_0(M/\mathcal{F}) = C^2_0(M/\mathcal{F})/E^2_0(M/\mathcal{F})\]

(2.5)

Now we define a map between the space \(\Omega^1(M/\mathcal{F})\) of basic 1-forms on \(M\) and the space \(J^2_0(M/\mathcal{F})\) of fields of basic 2-jets on \(M\). We show that this map induces an isomorphism between the cohomology group \(H^{2,2}_0(M/\mathcal{F})\) defined in (2.5) and the 1-dimensional basic cohomology group \(H^1(M/\mathcal{F})\) of \(M\).

Let \(\{dx^a, \theta^a\}\) be an adapted cobasis (1.5) on \(M\). To every basic 1-form \(\omega\) locally given
by $\omega = \omega_3 dx^3$, we associate
\[ D^3 \omega = \omega_3 \cdot j^1_3 dx^3 + \frac{1}{4} (X_{a_1} \omega_{a_2} + X_{a_2} \omega_{a_1}) j^2_3 dx^{a_1} \cdot j^2_3 dx^{a_2} \] (2.6)
Taking into account the transformation rule for the components of a basic 1-form at the
local chart changing, by a direct calculation we obtain that the coefficient functions of $D^3 \omega$
satisfy the relations (1.2), so (2.6) is a field of 2-jets which is basic because the functions $\omega_3$
are basic. The map $D^3 : \Omega^1(M/F) \rightarrow \mathcal{J}^2(M/F)$ defined by (2.6) is a monomorphism.

**Theorem 2.1.** Let $M$ be a Riemannian foliated manifold. The cohomology group $H^2_3(M/F)$
of fields of basic 2-jets on $M$ and the basic cohomology group $H^1(M/F)$ are isomorphic.

**Proof.** We denote by $Z^1(M/F)$ and $B^1(M/F)$ the sets of $d$-closed and $d$-exact basic
1-forms, respectively. We have the relations:
\[ D^4 (Z^1(M/F)) = C^2_0(M/F); \quad D^4 (B^1(M/F)) = E^2_0(M/F) \] (2.7)
Indeed, let be $\theta \in Z^1(M/F)$; from Poincaré lemma for $d$ we have that for any open
subset $V$ of $M$ there is $f_V \in \Phi(V)$ such that $\theta |_V = df_V = (X_a f_V) dx^a = \frac{\partial f_V}{\partial x^a} dx^a$. The
last equality is because $f$ is a basic function. Taking into account the definition of the
map $D^4$, we have $D^4 \theta |_V = \lambda^{1,2} f_V$ and then $D^4 (Z^1(M/F)) \subset C^2_0(M/F)$. Now let $\omega \in C^2_0(M/F)$
which means that for an open subset $V$ there is $f_V \in \Phi(V)$ such that $\omega |_V = j^{1,2} f_V$. A
simple calculation proves that the basic 1-form locally given by $\theta |_V = df_V$ is globally defined
on $M$ and satisfies the relations $d\theta = 0$ and $\omega = D^4 \theta$, hence $C^2_0(M/F) \subset D^4(Z^1(M/F))$. So,
the first equality is proved. An analogous argument is used for the second part of
the relation (2.7).

The relation (2.7) shows that $D^4$ induces an isomorphism between the quotient groups
$Z^1(M/F)/B^1(M/F)$ and $C^2_0(M/F)/E^2_0(M/F)$. Hence the announced result is proved. \(\blacksquare\)

## 3 Fields of leafwise 2-jets

Let $\mathcal{F}$ be a foliation of dimension $m$ and codimension $n$ on the Riemannian manifold
$(M, g)$.

**Definition 3.1.** Two maps $f, g \in \Omega^p(M)$ are said to determine the same (homogeneous)
leafwise 2-jet at $x \in M$ if they define the same (homogeneous) 2-jet at $x$ in the leaf
$L$ through $x$.

**Remark 3.1.** (a) If $f, g \in \Omega^p(M)$ determine the same 2-jet at $x$, then they determine the
same leafwise 2-jet at $x$. Indeed, if we consider $L$ to be the leaf through $x$ and $i : L \rightarrow M$
the inclusion of submanifold $L$ in $M$, for every curve $\sigma : R \rightarrow L$ with $\sigma(0) = x$, it is easy to
prove that the functions $f \circ i$ and $g \circ i$ determine the same 2-jet at $x$ on $L$.

(b) The reverse of (a) is not true. Indeed, for every basic function $h$, the functions $f$
and \( f + h \) determine the same leafwise 2-jet at a point \( x \), but they do not determine the same 2-jet on \( x \).

Let \( f, g \) be two functions on \( M \) which determine the same leafwise 2-jet at \( x \). From the definition, the maps \( f, g \) are zero at \( x \) and, for a distinguished chart \( U \) with local coordinates \( (x^a, x^\nu)_{a=n+1, \ldots, n+m} \), in \( U \cap \mathcal{L} \) they are satisfying:

\[
\frac{\partial f}{\partial x^a}(x) = \frac{\partial g}{\partial x^a}(x); \quad \frac{\partial^2 f}{\partial x^a \partial x^\nu}(x) = \frac{\partial^2 g}{\partial x^a \partial x^\nu}(x)
\]

The relation to "determine the same leafwise 2-jet at \( x \)" is an equivalence on \( \Omega^2(M) \) and an equivalence class \( j^L_2 f \) is called the leafwise 2-jet of \( f \) at \( x \).

We define the multiplication with a scalar \( \alpha \) of a class and the product of two classes by

\[
\alpha j^L_2 f = j^L_2(\alpha f); \quad j^L_2 f \cdot j^L_2 g = j^L_2(f \cdot g).
\]

We remark that \( j^L_2 x^a = 0 \) and that the classes \( j^L_2 f \) and

\[
\frac{\partial f}{\partial x^a}(x) j^L_2 x^a + \frac{1}{2} \frac{\partial^2 f}{\partial x^a \partial x^\nu}(x) j^L_2 x^a \cdot j^L_2 x^\nu \tag{3.1}
\]

are equal, hence \( (3.1) \) represent the local expression of \( j^L_2 f \).

For two distinguished charts with domains \( U \) and \( \overline{U} \) with \( U \cap \overline{U} \neq \emptyset \), from the Definition 3.1 we deduce the following relations

\[
j^L_2 x^a = \frac{\partial x^a}{\partial x^\nu}(x) j^L_2 x^\nu + \frac{1}{2} \frac{\partial^2 x^a}{\partial x^\nu \partial x^\nu}(x) j^L_2 x^\nu \cdot j^L_2 x^\nu
\]

\[
j^L_2 x^\nu \cdot j^L_2 x^\mu = \frac{\partial x^\nu}{\partial x^\mu}(x) \frac{\partial x^\mu}{\partial x^\nu}(x) j^L_2 x^\nu \cdot j^L_2 x^\mu \tag{3.2}
\]

where \( (x^a, x^\nu) \) are the local coordinates in the chart with domain \( \overline{U} \). From the Remark 3.1 we have \( j^L_2 f \subset j^L_2 f \).

Generally, we say that \( \omega^2_2 \) is a leafwise 2-jet at \( x \) on \( M \) if it has the following expression in a distinguished chart:

\[
\omega^2_2 = \omega^2 a(x) \cdot j^L_2 x^a + \frac{1}{2} \omega^2 a(x) \cdot j^L_2 x^a \cdot j^L_2 x^a \tag{3.3}
\]

where the coefficients \( \omega^2 a \), \( \omega^2 b \) are differentiable functions on \( M \) satisfying in \( U \cap \overline{U} \cap \mathcal{L} \)

\[
\omega^2 a = \omega^2 b \quad \text{and}
\]

\[
\omega^2 a \cdot \omega^2 b = \frac{\partial x^a}{\partial x^\nu} \frac{\partial x^b}{\partial x^\nu} + \omega^2 a \cdot \omega^2 b \frac{\partial^2 x^a}{\partial x^\nu \partial x^\nu} \tag{3.4}
\]

We denote by \( J^L_2(M) \) the space of leafwise 2-jets on \( M \). \( J^L_2(M) = \bigcup_{x \in M} J^L_2(M) \) is a fiber bundle over \( M \) with the fiber of dimension \( \frac{m(n+1)}{2} \). The sections in this bundle are denoted by \( J^2(F) \) and are called fields of leafwise 2-jets on \( M \).

The map \( j^L_2 : \Omega^2(M) \to J^2(F) \) which assigns to a function \( f \) its field of leafwise 2-jets is called the leafwise second differential on \( M \). This map is an algebras homomorphism.
From the relation (3.1) we obtain the local expression of the field of leafwise 2-jets of $f$:

$$j^{1,2} f = \frac{\partial f}{\partial x^v} j^{1,2} x^u + \frac{1}{2} \frac{\partial^2 f}{\partial x^v \partial x^w} j^{1,3} x^u \cdot j^{1,2} x^w \cdot j^{1,3} x^w$$  \hspace{1cm} (3.5)

**Remark 3.2.** Let $(M', F')$ be another Riemannian foliated manifold and $J^2(F')$ the space of fields of leafwise 2-jets on $M'$. If $\mu : M \to M'$ is a foliated map, then we have the map $\mu_0 : \Omega^0(M') \to \Omega^0(M)$ defined by the relation $\mu_0(f) = f \circ \mu$.

For $x \in M$ let $U, (x^u, x^v)$ be a distinguished chart at $x$ and $V, (y^k, y^l)$ be a distinguished chart at $\mu(x)$. We can associate to a field $j^{1,2} f \in J^2(F')$ the field $j^{1,2} (f \circ \mu) \in J^2(F)$; more generally, to a field $\omega^2 \in J^2(F)$ locally given by $\omega^2 = \omega'_u \cdot j^{1,2} y^u + \frac{1}{2} \omega''_v \cdot j^{1,2} y^v \cdot j^{1,2} y^v$, we associate the field $\mu^{1,2} (\omega^2) \in J^2(F)$ locally given by

$$\mu^{1,2} (\omega^2) = \omega'_u \cdot \frac{\partial y^u}{\partial x^v} \cdot j^{1,2} x^u + \frac{1}{2} \left( \omega''_v \frac{\partial y^u}{\partial x^v} \frac{\partial y^v}{\partial x^w} + \omega'_u \frac{\partial^2 y^u}{\partial x^v \partial x^w} \right) \cdot j^{1,2} x^u \cdot j^{1,2} x^w$$

We obtained in this way a map $\mu^{1,2} : J^2(F') \to J^2(F)$. If moreover $\mu$ is a diffeomorphism, then $\mu^{1,2}$ becomes an isomorphism.

**Definition 3.2.** The field of leafwise 2-jets $\omega \in J^2(F)$ is called:

(i) *exact* if $\omega = j^{1,2} f$ for some $f \in \Omega^1(M)$;

(ii) *closed* if it is locally exact.

We denote by $E^2(F)$ and $C^2(F)$ the spaces of exact, respectively closed fields of leafwise 2-jets on $M$ and we have $E^2(F) \subset C^2(F)$. We call the *cohomology group of fields of leafwise 2-jets on $M$* the following quotient group:

$$H^2_2(F) = C^2(F) / E^2(F)$$  \hspace{1cm} (3.6)

Now we define a map between the space $\Omega^{0,1}(M)$ of $(0,1)$-forms on $M$ and the space $J^2(F)$ of fields of leafwise 2-jets on $M$. Let (1.5) be an adapted cobasis. To every $(0,1)$-form $\omega$ locally given by $\omega = \omega_\nu \theta^\nu$, we associate the expression

$$D \omega = \omega_\nu \cdot j^{1,2} x^\nu + \frac{1}{4} \left( \frac{\partial \omega_\nu}{\partial x^u} + \frac{\partial \omega_u}{\partial x^\nu} \right) j^{1,2} x^u \cdot j^{1,2} x^v$$  \hspace{1cm} (3.7)

Taking into account the transformation rule for the components of a $(0,1)$-form at the local chart changing, by a direct calculation we obtain that the coefficient functions of $D \omega$ are satisfying the relations (3.4). So (3.7) is a field of leafwise 2-jets. Moreover the map $D : \Omega^{0,1}(M) \to J^2(F)$ defined by (3.7) is a monomorphism.

**Theorem 3.1.** Let $M$ be a Riemannian foliated manifold. The cohomology group of fields of leafwise 2-jets on $M$ and the Čech cohomology group $H^2(M, \Phi)$ are isomorphic.

**Proof.** We denote by $Z^{01}(M)$ and $B^{01}(M)$ the sets of $d_{01}$-closed and $d_{01}$-exact $(0,1)$-forms, respectively. We have the relations:

$$D(Z^{01}(M)) = C^2(F), \quad D(B^{01}(M)) = E^2(F)$$  \hspace{1cm} (3.8)
Indeed, let be $\theta \in \Omega^1(M)$; from Poincaré lemma [7] for $d_{01}$ we have that for an open subset $V$ of $M$ there is $f_V \in \Omega^0(V)$ such that $\theta|_V = d_{01}f_V = \frac{\partial f_V}{\partial x} \theta^x$. Taking into account the definition of the map $D$, we have $D\theta|_V = j^{1,0}f_V$ and then $D(\Omega^1(M)) \subset \Omega^2(F)$. Now, let $\omega \in \Omega^2(F)$ which means that for an open subset $V$ there is $f_V \in \Omega^0(V)$ such that $\omega|_V = j^{1,0}f_V$. A simple calculation proves that the $0.1$-form locally given by $\theta|_V = \frac{\partial f_V}{\partial x} \theta^x$ is globally defined on $M$ and satisfies the relations $d_{01}\theta = 0$ and $\omega = D\theta$, hence $\Omega^2(F) \subset D(\Omega^1(M))$.

So, the first equality (3.8) is proved. An analogous argument is used for the second part of the relation (3.8). The relation (3.8) shows that $D$ induces an isomorphism between the quotient groups $\Omega^1(M)/\delta \Omega^0(M)$ and $\Omega^2(F)/\delta \Omega^2(F)$. From the definitions for $H^0_2(M)$ and $H^2_2(F)$ and from the Theorem 1.1, the announced result is proved.

References


